Master’s Analysis Exam – January, 2015

3 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that a bounded sequence of real numbers has a convergent subsequence.

2. If \( \{a_n\} \) is a Cauchy sequence of real numbers, and \( f : \mathbb{R} \to \mathbb{R} \) is continuous, it is true that the sequence \( f(a_n) \) is Cauchy?

3. Let \( A, B \) be nonempty subsets of \( \mathbb{R}^n \). Prove that \( A \times B \) (in \( \mathbb{R}^{2n} \)) is an open set if both \( A \) and \( B \) are open.

4. Suppose \( X \) and \( Y \) are metric spaces and \( f_n : X \to Y \). Prove that if \( f_n \to f \) uniformly and each \( f_n \) is continuous, then \( f \) is continuous.

5. Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) have continuous first order partial derivatives and suppose that \( U \) is open. Prove that for any two points \((x, y)\) and \((a, b)\) in \( U \) sufficiently close together,

\[
f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b)
\]

and that \( \epsilon_1, \epsilon_2 \to 0 \) as \((x, y) \to (a, b)\).

6. If \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \), where \( U \) is open, and \( f \) has a local minimum at \((a, b) \in U \). Prove that \( \nabla f(a, b) = 0 \).

7. Use Green’s theorem to evaluate the line integral

\[
\int_{\Gamma} xy \, dx + y^2 \, dy,
\]

where \( \Gamma \) is the perimeter of the rectangle \([0, 4] \times [0, 1]\) oriented in the positive direction.

8. An integral in \( x, y, z \) space was converted to \( \iiint_E \rho^3 \sin \phi \, dV \) using spherical coordinates. What was the original function \( f(x, y, z) \) being integrated?

9. Let \( f \) be analytic in a region \( \Omega \) and \( a \in \Omega \). If \( f^{(\nu)}(a) = 0 \) for \( \nu = 0, 1, 2, 3, \ldots \), prove that \( f(z) = 0 \) for all \( z \in \Omega \).

10. Find the exact value of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).