Master's Analysis Exam – February 2016

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that the open interval $(0, 1)$ of $\mathbb{R}$ is uncountable.

2. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

   and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}.$$

   Give examples to show that each of these inequalities can be either equalities or strict inequalities.

3. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

4. Give an example of a sequence $\{a_n\}$ of positive real numbers such that $a_n \to 0$ but $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges.

5. Use the Inverse Function Theorem to give an implicit description of the points (x, y) in $\mathbb{R}^2$ where we can solve $u = x^4 + y^4/x$, $v = \sin x + \cos y$ for x, y in terms of u, v.

6. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an *even function* [that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an *odd function* [that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.

7. Prove the mean value theorem for integrals: if a function $f : [a, b] \to \mathbb{R}$ is continuous, then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a}\int_a^b f = f(x_0).$$

8. For a rectangle D in $\mathbb{R}^2$, let $f : D \to \mathbb{R}$ be continuous and nonnegative. Prove that if $\int_D f \, dA = 0$, then $f = 0$ on D.

9. Find the Laurent series for $f(z) = z/(z^2 + 1)$ about $z_0 = i$.

10. Show that the function $F(z) = |z^2 - z|$ is nowhere analytic.