Master's Analysis Exam – February 2017

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $n-1 \leq x \leq n$.

2. Assume a sequence $\{a_n\}$ of real numbers is bounded. Prove that if every convergent subsequence of $\{a_n\}$ converges to the same limit $a \in \mathbb{R}$, then $\{a_n\}$ converges to a.

3. Prove that if $f : [a,b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq M|x-y|^2$ for all $x, y \in [a,b]$, then f is a constant function.

4. Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\mathbf{u}) \geq ||\mathbf{u}||$ for every $\mathbf{u}$ in $\mathbb{R}^n$. Prove that $f^{-1}([0,1])$ is sequentially compact.

5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in M$, then f is uniformly continuous on $\mathbb{R}$.

6. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

7. Evaluate $\oint_C f\, dy - g\, dx$ where $f = 0$, $g = xy$ and C is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,4)$ with counterclockwise orientation.

8. Evaluate $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2} dx dy$.

9. For a nonempty open set A in $\mathbb{C}$, prove that if $f : A \to \mathbb{C}$ is analytic, then Re f and Im f are harmonic on A.

10. Let C be the ellipse $x^2/4 + y^2/9 = 1$ traversed once in the positive direction, and defines
$$G(z) := \int_C \frac{\xi^2 - \xi + 2}{\xi - z} d\xi \quad (z \text{ inside } C).$$
Find $G(1)$, $G'(i)$, and $G''(-i)$.