Master's Analysis Exam – September 2016

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges.

2. For each $n \in \mathbb{N}$ let A_n be a countable subset of $\mathbb{R}$. Prove that $\cup_{n=1}^{\infty} A_n$ is countable.

3. Suppose that (X, d) is a metric space that contains the point p and r is a positive number. Prove that the set $\{q \in X | d(p, q) \leq r\}$ is closed in X.

4. Prove the existence of an open cover of $\mathbb{R}^n$ that has no finite subcover.

5. For a pair of numbers a and b, consider the system of nonlinear equations
$$x + x^2 \cos y + xye^{x^3y^2} = a,$$
$$y + x^5 + y^3 - x^2 \cos(xy) = b.$$
Use the inverse function theorem to show that there is some positive number r such that if $a^2 + b^2 < r^2$, then this system of equations has at least one solution.

6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Prove that the gradient $\nabla f(x)$ is the direction along which f is increasing the fastest.

7. Let $\{f_n\}$ be a sequence of real-valued Riemann-integrable functions on the same compact interval $[a, b]$. Prove that if $\{f_n\}$ converges uniformly to a real-valued Riemann-integrable function f on $[a, b]$, then
$$\int_a^b f_n \to \int_a^b f.$$

8. Evaluate
$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2} dy\, dx.$$

9. Give two Laurent series expansions in powers of z for the function
$$f(z) = \frac{1}{z^2(1-z)},$$
and specify the regions in which those expansions are valid.

10. For a nonempty open set A in $\mathbb{C}$, prove that if $f : A \to \mathbb{C}$ is analytic and f' is continuous on and inside a simple closed curve γ lying inside A, then
$$\int_\gamma f = 0.$$