Master’s Analysis Exam – September 2016

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. Let \( x_n := \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \) for each \( n \in \mathbb{N} \). Prove that \( (x_n) \) is increasing and bounded, and hence converges.

2. For each \( n \in \mathbb{N} \) let \( A_n \) be a countable subset of \( \mathbb{R} \). Prove that \( \bigcup_{n=1}^{\infty} A_n \) is countable.

3. Suppose that \( (X,d) \) is a metric space that contains the point \( p \) and \( r \) is a positive number. Prove that the set \( \{ q \in X | d(p,q) \leq r \} \) is closed in \( X \).

4. Prove the existence of an open cover of \( \mathbb{R}^n \) that has no finite subcover.

5. For a pair of numbers \( a \) and \( b \), consider the system of nonlinear equations
   \[
   \begin{align*}
   x + x^2 \cos y + xy e^{x^3 y^2} &= a, \\
   y + x^5 + y^3 - x^2 \cos(xy) &= b.
   \end{align*}
   \]

   Use the inverse function theorem to show that there is some positive number \( r \) such that if \( a^2 + b^2 < r^2 \), then this system of equations has at least one solution.

6. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable. Prove that the gradient \( \nabla f(x) \) is the direction along which \( f \) is increasing the fastest.

7. Let \( \{ f_n \} \) be a sequence of real-valued Riemann-integrable functions on the same compact interval \([a,b]\). Prove that if \( \{ f_n \} \) converges uniformly to a real-valued Riemann-integrable function \( f \) on \([a,b]\), then
   \[
   \int_a^b f_n \to \int_a^b f.
   \]

8. Evaluate
   \[
   \int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2} dy \, dx.
   \]

9. Give two Laurent series expansions in powers of \( z \) for the function
   \[
   f(z) = \frac{1}{z^2(1-z)},
   \]
   and specify the regions in which those expansions are valid.

10. For a nonempty open set \( A \) in \( \mathbb{C} \), prove that if \( f : A \to \mathbb{C} \) is analytic and \( f' \) is continuous on and inside a simple closed curve \( \gamma \) lying inside \( A \), then
    \[
    \int_{\gamma} f = 0.
    \]