Master’s Analysis Exam – September 2015

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. For a bounded sequence \( \{a_n\} \) of real numbers, let \( L \) be the set of limits of convergent subsequences of \( \{a_n\} \). Prove that if \( \sup L = \inf L \), then \( \{a_n\} \) converges.

2. Let \( \{b_n\} \) be a bounded sequence of nonnegative numbers and let \( r \) be any number such that \( 0 \leq r < 1 \). Define

\[
s_n = b_1r + b_2r^2 + \cdots + b_n r^n \quad \text{for every } n \geq 1.
\]

Prove the sequence \( \{s_n\} \) converges.

3. For a compact interval \( C \) of \( \mathbb{R} \), let \( f : C \to \mathbb{R} \) be continuously differentiable. Prove that \( f \) is Lipschitz on \( C \).

4. Prove that if \( K_1 \) and \( K_2 \) are sequentially compact subsets of \( \mathbb{R}^n \) then there exists points \( x_1 \in K_1 \) and \( x_2 \in K_2 \) such that if \( z_1 \in K_1 \) and \( z_2 \in K_2 \), then \( \|z_1 - z_2\| \geq \|x_1 - x_2\| \).

5. Let \( f : [a, b] \to [a, b] \) be differentiable. Show that \( f \) is a contraction of \([a, b]\) if and only if there exists \( k < 1 \) with \( |f'(x)| \leq k \) for all \( x \in (a, b) \).

6. Give an example of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) whose partial derivatives exist at \((0, 0)\), but for which \( f \) is not differentiable at \((0, 0)\).

7. Evaluate

\[
\iint_S (x - y)^2 \sin^2(x + y) \, dx \, dy
\]

where \( S \) is the parallelogram with vertices \((\pi, 0), (2\pi, \pi), (\pi, 2\pi), (0, \pi)\).

8. Let \( D \) be a region in \( \mathbb{R}^2 \) for which Green’s Theorem applies, and let \( C \) be the boundary curve of \( D \). Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) is twice continuously differentiable. Prove that if \( \partial^2 f / \partial x^2 + \partial^2 f / \partial y^2 = 0 \) on \( D \), then \( \int_C \partial f / \partial x \, dx - \partial f / \partial y \, dy = 0 \).

9. Suppose \( f : \mathbb{C} \to \mathbb{C} \) is complex analytic. Prove that \( g(z) = \overline{f(\overline{z})} \) is complex analytic.

10. Evaluate the integral

\[
\int_0^\infty \frac{1}{(1 + x^2)x^{1/2}} \, dx.
\]