MASTER'S ANALYSIS EXAM – AUGUST 2012

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{1+n^3 x}$ converges uniformly or only pointwise on $[0.2\pi]$.

2. Let (a_n) be a decreasing sequence of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $na_n \to 0$ as $n \to \infty$.

3. Let (X, d_X), (Y, d_Y) be metric spaces and $p \in X$. Prove that the following statements for a mapping $f : X \to Y$ are equivalent

   (a) If (p_n) is a sequence in X converging to p, then the sequence $f(p_n)$ converges to $f(p)$.

   (b) For every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(f(p), f(q)) < \epsilon$ for each $q \in X$ with $d_X(p, q) < \delta$.

4. Let U be an unbounded open set in $(1, \infty)$. Determine whether $f(x) = \log(x)$ is uniformly continuous on U.

5. Let (X, d) be a nonempty complete metric space. Let f be a function mapping X to X. Suppose for every $x, y \in X$, there is a number $\lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

   Show that the set $\{x \in X : \; f(x) = x\}$ has exactly one element. (Note: you need to provide more details than just quoting the contraction mapping principle or any comparable result)

6. Prove that the function

$$f(x, y) = \left(\frac{-y}{\sqrt{1 + x^2 + y^2}}, \frac{x}{\sqrt{1 + x^2 + y^2}} \right)$$

   has an inverse in the neighborhood of every point of $\mathbb{R}^2$. Find the explicit form of the inverse of f.

7. Prove that if $f : [a, b] \to \mathbb{R}$ is monotone, then f is Riemann integrable.

8. Prove the following mean value theorem for integrals: if functions f and g are defined and continuous and g is non-negative over interval $[a, b]$, then there is a point $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx.$$

9. Compute the integral $\int_0^{\infty} \frac{1}{1 + x^{2n}}\,dx$ for $n \geq 1$. (Hint: use the boundary of a sector with angle π/n as a contour.)

10. Let $f(z) = \log(1 + z)$ be the branch of the logarithm with $-\pi < \arg z < \pi$. Determine the Taylor series for $f(z)$ around $z = 0$ and give its radius of convergence.