# MASTER'S ANALYSIS EXAM – FEBRUARY 2012

1. Consider the sequence $a_1 = 1,\ a_{n+1} = \sqrt{a_n + 2}$. Show that $(a_n)$ is monotone and converges to 2.

2. Determine whether the following series is absolutely convergent, conditionally convergent or divergent for various values of $\alpha$

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^2 e^{-n}$  (b) $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{n} - \alpha)^{n/2}$  (c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\alpha n}{n+4}\right)^n$

3. Prove that $f(x) = x^\alpha,\ 0 < \alpha < 1$, is uniformly continuous on $[0, \infty)$.

4. Prove that (a) every Cauchy sequence is bounded and that (b) a Cauchy sequence is convergent if it has a convergent subsequence.

5. Find the $n$th degree Taylor polynomial for the function $f(x) = \ln(\sqrt{2+x})$ at $x = 0$, with $f$ mapping $(-2, \infty)$ to $\mathbb{R}$.

6. Find the maximum value of $x + 2y + 4z$ subject to the condition $x^2 + 6y^2 - z^2 + 9 = 0$.

7. Let $f$ be Riemann integral in $[a, b]$ and let
   $$\omega(x, p) = \sup\{f(y) : y \in (x-p, x+p)\} - \inf\{f(y) : y \in (x-p, x+p)\}.$$
   For any $c > 0$ and $\epsilon > 0$, show that the set
   $$E = \{x \in [a, b] :\ \omega(x, p) > c \text{ for all } p > 0\}$$
   can be covered with closed intervals of total length less than $\epsilon$.

8. Integrate the function $xe^{-x^2(1+y^2)}$ over the set $(0, \infty) \times (0, \infty)$. By reversing the order of integration, deduce that
   $$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

9. Find the Laurent series for the function $\dfrac{z}{(z+1)(z-2)}$ in the domain

   (a) $|z| < 1$
   (b) $1 < |z| < 2$.

10. Let $m > 0$, and let $P$, $Q$ be polynomials such that
    $$\deg Q \geq 1 + \deg P.$$
    Show that
    $$\lim_{\rho\to\infty} \int_{C_\rho} e^{imz}\,\frac{P(z)}{Q(z)}\,dz = 0,$$
    where $C_\rho$ is the the semicircle of radius $\rho$ from $\rho$ to $-\rho$ in the upper half plane.