Master's Analysis Exam – February, 2014

4 Hours. No notes, textbooks, or calculators.

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that the sequence $f_n(x) = \dfrac{\sin(nx)}{n}$ converges uniformly to 0 on $\mathbb{R}$.

2. Define a metric d on $C[0,1]$ (continuous real-valued functions on $[0,1]$) by
$$d(f,g) = \sup_{0\le x\le 1} |f(x) - g(x)|.$$
Prove or disprove: the sequence $f_k(x) = x^k$ Cauchy in this metric.

3. Suppose f is a continuous mapping from a compact metric space X to a metric space Y. Prove that $f(X)$ is sequentially compact.

4. Prove or disprove: the interior of $A \subset \mathbb{R}^n$ is open.

5. Prove that the function
$$g(x) = \begin{cases} x^3 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
is differentiable on $\mathbb{R}$ and that $g'(x)$ continuous on $\mathbb{R}$.

6. Suppose the first-order partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at every point in $\mathbb{R}^n$. If f has a local minimum at $x_0 \in \mathbb{R}^n$ (i.e., there exists $\delta > 0$ such that $f(x_0 + h) \ge f(x_0)$ for all h satisfying $\|h\| < \delta$), then $\nabla f(x_0) = 0$.

7. Prove there exists a sequence of real-valued continuous functions $f_k(x)$ defined on $[0,1]$ for which $f_k(x)$ converges pointwise to a continuous function $f(x)$ defined on $[0,1]$, but for which $\int_0^1 f_k(x)dx = 1$ for all $k \in \mathbb{N}$ while $\int_0^1 f(x)dx = 0$.

8. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Prove that
$$2\int_a^b \left(f(x) \int_x^b f(y)dy \right) dx = \left(\int_a^b f(x)dx \right)^2.$$

9. Suppose $f(z)$ is complex analytic on an open connected set A of $\mathbb{C}$. Prove that if $|f(z)|$ is constant in A, then $f(z)$ is constant in A.

10. Find the maximum value of $|\sin(z)|$ on the square $0 \le \mathrm{Re}(z) \le 2\pi$, $0 \le \mathrm{Im}(z) \le 2\pi$.