Master's Analysis Exam – January, 2014

3 Hours. No notes, textbooks, or calculators.

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that the sequence $f_n(x) = x/(nx+1)$ converges uniformly to some function $g(x)$ on $[0, 1]$.

2. Prove that if $\{f_n\}$ is a sequence of continuous functions defined on an interval I that converges uniformly to a function f defined on I, then f is continuous.

3. Let f be a continuous mapping of a metric space X into a metric space Y. Prove that if E is a dense subset of X, then $f(E)$ is dense in $f(X)$.

4. Prove or disprove: If $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and A is a closed subset of $\mathbb{R}^n$, then $f(A)$ is a closed subset of $\mathbb{R}^m$.

5. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and $f : \mathcal{O} \to \mathbb{R}$ a function such that for some $x \in \mathcal{O}$ there is a vector $A_x \in \mathbb{R}^n$ satisfying

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle A_x, h \rangle}{\|h\|} = 0$$

where $\langle A_x, h \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Prove that the first-order partial derivatives of f exist at x and that $A_x = \nabla f(x)$.

6. Consider the mapping $F(r, \theta) = (r \cos\theta, \sin\theta)$, $(r, \theta) \in \mathbb{R}^2$. At what points is the Inverse Function Theorem applicable to F?

7. For the curve Γ parameterized by $\Gamma(t) = (t, kt)$, $0 \le t \le 1$, evaluate the line integral

$$\int_\Gamma x \sin y dx + y \cos x dy.$$

8. Use Darboux sums to prove that if f and g are Riemann integrable real-valued functions on a generalized rectangle I in $\mathbb{R}^n$, then so is $f + g$ and

$$\int_I f + \int_I g = \int_I (f + g).$$

9. For $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, if $f'(z)$ exists, prove that $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.

10. For γ the unit circle in $\mathbb{C}$, use Cauchy's Integral Formula to compute

$$\int_\gamma \frac{\cos z}{z} dz.$$