MASTER'S EXAM, ALGEBRA, AUGUST 2012

1. Let V be the vector space of polynomials in x over $\mathbb{C}$ of degree less than 5. Let $T : V \to V$ be $T(f) = f'' - 6f' + 9f$. Show that T is a linear transformation and find its kernel.

2. Prove that if the eigenvalues of a diagonalizable matrix are all ± 1, then the matrix is its own inverse.

3. Find a symmetric matrix similar to $\left[\begin{smallmatrix} 1 & -2 \\ 1 & 3 \end{smallmatrix}\right]$ or prove that such a matrix does not exist.

4. Show that every index 2 subgroup of a group is normal. Give an example (with proof) of a group with a non-normal index 3 subgroup.

5. Given an action of G on a set, show that every point of some orbit has the same stabilizer if and only if this stabilizer is a normal subgroup of G.

6. Prove that if G is a cyclic group and N is a normal subgroup of G, then G/N is a cyclic group.

7. Let R be a commutative ring with 1 and with elements a, b of additive orders 5 and 7 respectively. Is R an integral domain?

8. If I is an ideal of the ring R, prove that $I[x]$ is an ideal in the polynomial ring $R[x]$ and that $R[x]/I[x] \cong (R/I)[x]$.

9. Let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive 5th root of unity.

   (a) Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order 4.

   (b) Find (with proof) a polynomial $g(x) \in \mathbb{Q}[x]$ such that the unique quadratic subfield of $K/\mathbb{Q}$ is $Q(\alpha)$, where α is a root of $g(x)$.

10. Determine the minimal polynomial over $\mathbb{Q}$ of the element $\beta = \sqrt{2} + \sqrt{5}$.