MASTER'S EXAM, ALGEBRA, JANUARY 2013

1. Find a 3×3 orthogonal matrix with all entries nonzero. (No partial credit.)

2. Find the dimension of the space of $n \times n$ matrices with trace 0. Justify your answer. (The trace of a square matrix is the sum of its diagonal entries.)

3. Let V and W be finite-dimensional vector spaces, and let $L : V \to W$ be a linear transformation. Prove that $dim(ker(L)) + dim(image(L)) = dim(V)$.

4. Let p, q be primes. Show that a group of order pq is not simple.

5. Let Q be the quaternion group of order 8. Prove that every subgroup of Q is normal.

6. Let G be a group with center $Z(G)$. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

7. Let R be a commutative ring with 1. Let $x, y \in R$ and let I be the smallest ideal containing x and y. Show that $I = \{ax + by : a, b \in R\}$.

8. In a principal ideal domain, show that every nonzero prime ideal is maximal.

9. Calculate the 36th cyclotomic polynomial $\Phi_{36}(x)$.

10. Let F be a field with 125 elements. Does the polynomial $x^2 + x + 1$ have a root in F? (The polynomial should be considered as having coefficients in $\mathbb{Z}/5\mathbb{Z}$.) Justify your answer.