1 Analysis Exam Topics (Version June 09, 2010)

1. Analysis in $\mathbb{R}$ (2 questions)
   (a) Characterization as a complete, ordered field. Cardinality of subsets of $\mathbb{R}$.
   (b) Archimedean Property, convergence of bounded monotonic sequences
   (c) Convergence of sequences and series (algebraic rules, sequences and series, Cauchy criterion, common convergence tests, convergence of rearranged series, L'Hôpital's Rule)
2. Topological and Uniform Properties (2 questions)
   (a) Metric spaces and their properties, subspace topology
   (b) Open and closed sets (closure, limit points), boundedness
   (c) Limits and continuity (sequences, subsequences, convergence, continuous functions, algebraic rules of limits and continuous functions)
   (d) Cauchy sequences and completeness
   (e) Compactness (open coverings, sequential compactness, continuous maps of compact sets, finite-intersection property, Heine-Borel theorem, Bolzano-Weierstrass theorem, extreme-value theorem)
   (f) Connectedness (continuous maps of connected sets, Intermediate Value Theorem)
   (g) Uniform continuity, Lipschitz continuity, preservation of Cauchy sequences
   (h) Uniform convergence, convergence of sequences and series of functions, Weierstrass comparison test
   (i) Relation of uniform convergence to continuity, differentiation, and integration
3. Differentiation (2 questions)
   (a) Normed linear spaces, inner product spaces, Euclidean geometry of $\mathbb{R}^n$
   (b) Differentiation (definition as a linear transformation, differentiability, directional, partial, and total derivatives)
   (c) Product and quotient rules, Chain rule
   (d) Mean value theorem (Rolle's theorem in $\mathbb{R}$, integral version)
   (e) Higher-order derivatives, Taylor's theorem
   (f) Contraction Mapping Principle and its applications (Newton's method, successive approximations)
   (g) Inverse and implicit function theorems
   (h) The second derivative test and characterization of local extrema
   (i) Lagrange multipliers

4. Integration (2 questions)
   (a) Riemann-Darboux integral on $\mathbb{R}^n$
   (b) Basic properties of integration (integrability of continuous and monotonic functions, rules for combining and comparing integrals, the Fundamental Theorem of Calculus, improper integrals)
   (c) The Jacobian and change of variables (including spherical and cylindrical coordinates)
   (d) Iterated integrals and Fubini's theorem
   (e) Line and surface integrals
   (f) Green's theorem, Stokes' theorem, and Gauss' divergence theorem
5. Complex Variables (2 questions)
   (a) Analytic functions, Cauchy-Riemann equations, harmonic functions
   (b) Elementary functions in the complex plane, exponential and log functions, complex exponents, trigonometric & hyperbolic functions
   (c) Contour integrals, upper bounds, primitives, Cauchy-Goursat theorem, Cauchy integral formulas, Liouville's theorem, fundamental theorem of algebra, maximum modulus principle
   (d) Isolated singularities, Laurent series
   (e) Residue theorem, its application to improper integrals, Jordan's lemma
   (f) Argument principle, Rouché's theorem

References

[1] S. Abbott, *Understanding Analysis*, Springer-Verlag, New York, first ed., 2001.

[2] C.H. Edwards Jr., *Advanced Calculus of Several Variables*, Dover Publications, Inc., New York, revised ed., 1995.

[3] J.E. Marsden and M.J. Hoffman, *Basic complex analysis*, W. H. Freeman and Company, New York, third ed., 1998.

[4] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Co., New York, third ed., 1976.

[5] E.B. Saff and A.D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics*, Prentice Hall, New York, third ed., 2003.

2 Sample Questions

2.1 Analysis in $\mathbb{R}$

1. Show that $[0, 1]$ is uncountable.

2. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, show $\lim_{n\to\infty} a_n b_n = AB$. State clearly any assumptions.

3. Let $A = \{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 \leq a_2 \leq a_3 \leq \cdots$. If L is the least upper bound of the set A , show that $\lim_{n\to\infty} a_n = L$.

4. Prove that if $a_i \geq a_{i+1} \geq 0$, $\forall i$ and $(a_i) \to 0$, then $\displaystyle\sum_{i=1}^{\infty} (-1)^i a_i$ converges.

5. Show that a sequence of real numbers has a monotone subsequence. (A sequence x_n is monotone if either $x_n \leq x_{n+1}$ for each n or $x_n \geq x_{n+1}$ for each n).

2.2 Topological and Uniform Properties

1. Let A be a compact subset of $\mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^n$. Use the Extreme Value Theorem to prove that there is $\boldsymbol{a}_0 \in A$ such that $\|\boldsymbol{a}_0 - \boldsymbol{v}\| \leq \|\boldsymbol{a} - \boldsymbol{v}\|$ for any $\boldsymbol{a} \in A$. Is this point $\boldsymbol{a}_0$ unique?

2. Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x) - f(y)\| \leq k\|x - y\|$ for some constant $k \in [0, 1)$. Prove that f has a unique fixed point.

3. Prove that the uniform limit of continuous functions is continuous.

4. Let $f_n(x) = nxe^{-nx}$. Prove that $f_n \to 0$ pointwise, but not uniformly on $[0, 1]$, as $n \to \infty$.

5. Prove the intermediate value theorem, i.e., if $f : [a, b] \to \mathbb{R}$ is continuous and d is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.

2.3 Differentiation

1. If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), show that there is a number c, $a < c < b$ so that $f'(c)(b-a) = f(b)-f(a)$.

2. Prove that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is differentiable at 0.

3. Suppose f is differentiable on (a, b) and continuous on $[a, b]$ and $f(a) = f(b)$. Prove that if $f(c) > f(a)$ for some $c \in (a, b)$, then $\exists x_1, x_2 \in (a, b)$ such that $f'(x_1) > f'(x_2)$.

4. Find all extrema of $f(x, y, z, w) = 3x + y + w$ subject to the constraints $3x^2 + y + 4z^3 = 1$ and $-x^3 + 3z^4 + w = 0$.

5. Give a complete statement of the Implicit Function Theorem for a function $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^n$, where $m > n \geq 1$.

2.4 Integration

1. If f is a continuous function on $[a, b]$, show $\int_a^b f(x)dx = f(c)(b - a)$ for some $c,\ a \leq c \leq b$.

2. Let f be a bounded function on $[a, b]$. Show f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there is a partition P of [a,b] so that $U_f(P) - L_f(P) < \varepsilon$, where $U_f(P)$ and $L_f(P)$ denote the upper and lower sum of f with respect to the partition P.

3. Evaluate the following iterated integral $\int_0^{\sqrt{\frac{\pi}{2}}} \int_y^{\sqrt{\frac{\pi}{2}}} y^2 \sin x^2 dx\, dy$.

4. Find all values of p for which the improper integral $\int_0^1 \frac{1}{x^p}\, dx$ is finite.

5. Let S be the subset of $\mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

   Evaluate the surface integral $\int_S f\, d\sigma$ where $f(x, y, z) = z^2$.

2.5 Complex Variables

1. Let $f(z) = \sin z$. Find a point z_0 in the rectangular region $0 \leq x \leq \pi$, $0 \leq y \leq 1$ where $|f|$ assumes its maximum value. Show that there is only one such point z_0.

2. Suppose that $f(z)$ is entire and $\operatorname{Re} f(z)$ has an upper bound C in the complex plane. By considering e^f, show that f is constant.

3. Determine the number of roots of the equation

$$z^{11} - 4z^3 + z^2 - 1 = 0$$

   inside the circle $|z| = 1$.

4. Use the Residue Theorem to compute and simplify the improper integral $\int_0^\infty \frac{dx}{1 + x^{10}}$.

5. Suppose that f is entire and $\lim_{z\to\infty} \frac{f(z)}{z} = 0$. Prove that f is constant.