FALL 2012 - PH.D. PRELIMINARY EXAMINATION ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

(1) State and prove either the Cauchy-Peano Existence Theorem or the Picard-Lindelöf Existence and Uniqueness Theorem for initial value problems. You may assume the Arzelá-Ascoli Theorem and the Contraction Mapping Principle.

(2) Assume that $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous flow and the positive orbit of the point p is bounded. Show that the ω-limit set of p is invariant, closed, and connected.

(3) Answer the following.
   (a) Define a fundamental matrix to $\mathbf{x}' = A(t)\mathbf{x}$.
   (b) Prove that if $\int_1^\infty \mathrm{Tr}(A(t))\,dt = \infty$, then there exists a solution $\mathbf{x}(t)$ of $\mathbf{x}' = A(t)\mathbf{x}$ such that $|\mathbf{x}(t)|$ is unbounded for $t \geq 1$.

(4) Consider the equation $u'' + u' + g(u) = 0$ where g is continuously differentiable, $ug(u) > 0$ for $u \neq 0$, and $g(x) = bx + o(|x|)$. Show that the zero solution is asymptotically stable.

(5) Let A be a transition matrix for a subshift of finite type, and let Σ_A be the associated 2-sided subshift of finite type.
   (a) Prove that if there is an element $s \in \Sigma_A$ that contains a symbol i at least twice, then there is a periodic element $s' \in \Sigma_A$ such that $s'_0 = i$. Such a symbol is called *essential*.
   (b) Prove that any ω-limit point in Σ_A contains only essential elements.

(6) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and p be a hyperbolic saddle fixed point. Assume that q is a homoclinic point for p.
   (a) Prove that q is a chain recurrent point.
   (b) Prove that q is a nonwandering point provided that q is a transverse homoclinic point for p.

(7) Answer the following:
  (a) Define topological conjugacy.
  (b) State the Harman-Grobman Theorem.
  (c) Find a topological conjugacy between the following two matrices

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(8) Compute the topological entropy for the map $f : S^1 \to S^1$ defined by $f(x) = (2x)\text{mod}1$.