FALL 2012 - PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS AND
DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

1. State and prove either the Cauchy-Peano Existence Theorem or the Picard-Lindelöf Existence and Uniqueness Theorem for initial value problems. You may assume the Arzelà-Ascoli Theorem and the Contraction Mapping Principle.

2. Assume that $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous flow and the positive orbit of the point $p$ is bounded. Show that the $\omega$-limit set of $p$ is invariant, closed, and connected.

3. Answer the following.
   (a) Define a fundamental matrix to $x' = A(t)x$.
   (b) Prove that if $\int_1^\infty \text{Tr}(A(t)) \, dt = \infty$, then there exists a solution $x(t)$ of $x' = A(t)x$ such that $|x(t)|$ is unbounded for $t \geq 1$.

4. Consider the equation $u'' + u' + g(u) = 0$ where $g$ is continuously differentiable, $ug(u) > 0$ for $u \neq 0$, and $g(x) = bx + o(|x|)$. Show that the zero solution is asymptotically stable.

5. Let $A$ be a transition matrix for a subshift of finite type, and let $\Sigma_A$ be the associated 2-sided subshift of finite type.
   (a) Prove that if there is an element $s \in \Sigma_A$ that contains a symbol $i$ at least twice, then there is a periodic element $s' \in \Sigma_A$ such that $s'_0 = i$. Such a symbol is called essential.
   (b) Prove that any $\omega$-limit point in $\Sigma_A$ contains only essential elements.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and $p$ be a hyperbolic saddle fixed point. Assume that $q$ is a homoclinic point for $p$.
   (a) Prove that $q$ is a chain recurrent point.
   (b) Prove that $q$ is a nonwandering point provided that $q$ is a transverse homoclinic point for $p$. 
(7) Answer the following:
(a) Define topological conjugacy.
(b) State the Harman-Grobman Theorem.
(c) Find a topological conjugacy between the following two matrices
\[
\begin{bmatrix}
3 & 0 \\
0 & 2 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
2 & 0 \\
0 & 2 \\
\end{bmatrix}
\]

(8) Compute the topological entropy for the map \( f : S^1 \to S^1 \)
defined by \( f(x) = (2x) \mod 1 \).