QUALIFYING EXAM –Differential Equations, August 2010

Instructions: Give solutions to the exactly 6 of the following 7 problems.

1. Let a fundamental matrix solution $X(t)$ for a $n \times n$ system $x' = A(t)x$ satisfy

$$|X(t)X^{-1}(s)| \leq K \quad \text{for} \quad 0 \leq s \leq t < \infty,$$

where K is a positive constant. Let the $n \times n$ matrix-valued function B be continuous on $[0, \infty)$ and satisfy

$$\int_0^\infty |B(t)| dt < \infty.$$

Prove that $x = 0$ is uniformly stable for the equation

$$x' = (A(t) + B(t))x.$$

2. Consider the equation $x' = Ax + h(t)$ in $\mathbb{R}^n$, where A has n eigenvalues with negative real parts and h is continuous and bounded on $\mathbb{R}$. Prove that there is a unique solution bounded on $(-\infty, \infty)$.

3. Let $f(x)$ be a smooth bounded vector field on the plan.
(a) Prove that if $x' = f(x)$ has a periodic solution then it has a stationary solution.
(b) Suppose that $x' = f(x)$ has no stationary solutions. Prove that for any solution $x(t)$ defined on $[0, +\infty)$,

$$\lim_{t \to +\infty} |x(t)| = +\infty.$$

4. Prove the following instability theorem: Let V be a C^1 real-valued function defined on a neighborhood U of an equilibrium point $\bar{x}$ of $x' = f(x)$. Suppose $V(\bar{x}) = 0$ and $\dot{V}(x) > 0$ in $U - \bar{x}$. If there is a convergent sequence $x_n \to \bar{x}$ such that $V(x_n) > 0$, then $\bar{x}$ is unstable.

5. Find all equilibrium points for the following equation. Determine the stability of the equilibrium by find an appropriate Lyapunov function

$$x'' + (x^2 - 1)x' + x = 0.$$

6. Show that $\dot{x} = 2x + e^{-|x|} \sin\, t$ has a 2π-periodic solution.

7. Consider $x' = Ax + f(x), \quad x \in \mathbb{R}^n$, where A is a $n \times n$ matrix and f is a C^1 function with $f(0) = 0$ and $f'(0) = 0$. Prove that if A has an eigenvalue with positive real part, then the equilibrium point 0 is unstable.