Ph.D. Qualifying Exam: Applied Math/ODE
August/September 2016

Instructions: answer 6 of the 8 questions. If you answer more than 6 questions, the first 6 questions will be graded.

1. State and prove Gronwall's Inequality.

2. Give an example of $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, $(x,y) \in \mathbb{R}^2$, where the origin $(0,0)$ is an equilibrium whose linearization is a center, but nonlinearly the equilibrium at $(0,0)$ is asymptotically stable equilibrium. Be sure to prove the asymptotic stability of the equilibrium at $(0,0)$.

3. For an $n \times n$ real matrix A, suppose there are positive real constants $\lambda > 0$ and $\tau > 0$ such that

$$\|e^{tA}v\| \leq e^{-\lambda t}\|v\|$$

for all $t \geq \tau$ and for all $v \in \mathbb{R}^n$ (where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$). Define

$$\|v\|_a = \int_0^\tau e^{\lambda s}\|e^{sA}v\|ds.$$

Prove that $\|v\|_a$ satisfies

$$\|e^{tA}v\|_a \leq e^{-\lambda t}\|v\|_a$$

for all $v \in \mathbb{R}^n$ and for all $t \geq 0$.

4. Give a direct proof that the point $(1/\sqrt{2}, 1/\sqrt{2})$ on the unit circle is an ω-limit point of $(3,8)$ for the flow ϕ_t of

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2),$$

that is, find a strictly increasing positive sequence $\{t_n\}$ such that

$$\phi_{t_n}\big((3,8)\big) \to (1/\sqrt{2}, 1/\sqrt{2}).$$

5. Suppose continuous functions $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, where X and Y are compact metric spaces. Prove that if f is expansive, then g is expansive.

6. Suppose $f : X \to X$ is transitive where X is a compact metric space. Prove or disprove: the ω-limit set of any $x \in X$ is always the entire space X.

7. Prove that the topological entropy of a circle rotation $R_\alpha(x) = x + \alpha \bmod 1$ is zero for any $\alpha \in \mathbb{R}$.

8. Let $A \in \{0,1\}^{n\times n}$ be an eventually positive matrix, i.e., there is $K > 0$ such that $(A^K)_{ij} > 0$ for all $1 \leq i,j \leq n$.

   (i) Prove for all $k \geq K$ that $(A^k)_{ij} > 0$ for all $1 \leq i,j \leq n$.

   (ii) Prove that $\sigma_A : \Sigma_A^+ \to \Sigma_A^+$, the subshift of finite type for A, is topologically mixing.