# WINTER 2011 - PH.D. PRELIMINARY EXAMINATION
# ORDINARY DIFFERENTIAL EQUATIONS

(1) Consider the following differential equation in $\mathbb{R}^n$

$$x' = Ax + B(t)x + g(x,t).$$

Assume that

(a) $A$ is a constant $n \times n$ matrix with only eigenvalues with negative real parts,

(b) $B(t)$ is the $n \times n$ matrix, continuously dependent on $t$ such that $||B(t)|| \to 0$ as $t \to \infty$,

(c) $g(x,t)$ is $C^2$ and there are constants $a > 0$ and $k > 0$ such that $||g(x,t)|| \leq k||x||^2$ for all $t \geq 0$ and $||x|| < a$.

Prove that there are constants $C > 1$, $\delta > 0$, $\lambda > 0$ such that

$$||x(t,t_0,x_0)|| \leq C||x_0||e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

whenever $||x_0|| \leq \delta/C$, where $x(t,t_0,x_0)$ is the solution of above equation with $x(t_0,t_0,x_0) = x_0$.

(2) Prove that the following differential equations in $\mathbb{R}^2$

$$x' = x - y - x^3, \quad y' = x + y - y^3$$

has a unique globally attracting limit cycle on the punctured plane $\mathbb{R}^2 - \{0\}$.

(3) Consider a smooth differential equation on the plane

$$x' = g(x,y), \quad y' = h(x,y)$$

and let $f(x,y) = (g(x,y), h(x,y))$. If the divergence of $f$ given by

$$\text{div } f(x,y) = \frac{\partial}{\partial x} g(x,y) + \frac{\partial}{\partial y} h(x,y)$$

is not identically zero and of fixed sign in a simply connected region $\Omega$. Prove that the system has no periodic orbits in $\Omega$.

(4) Suppose that $V$ is a smooth function defined on an open neighborhood $U$ of an equilibrium point $\bar{x}$ of the differential equation $x' = f(x)$, where $f$ is a smooth function from $\mathbb{R}^n$ into itself, such that $V(\bar{x}) = 0$ and $\dot{V}(x) = (\text{grad } V(x), f(x)) > 0$ on $U - \{\bar{x}\}$. Assume that for each neighborhood $N$ of $\bar{x}$, there is a point $x^* \in N$ such that $V(x^*) > 0$. Prove $\bar{x}$ is unstable.

(5) Let $\Lambda$ be a hyperbolic attractor (so $\Lambda$ is a transitive hyperbolic set and there exists a neighborhood $U$ of $\Lambda$ such that $\bigcap_{n\geq 0} f^n(U) = \Lambda$).
   (a) Prove that if $x \in \Lambda$, then $W^u(x) \subset \Lambda$.
   (b) Assume in addition that $\Lambda$ is topologically mixing. Prove that if $p \in \text{Per}(\Lambda)$, then $W^s(p)$ is dense in $W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x)$.

(6) Let $f : X \to X$ be a continuous map of a compact metric space. Let $\overline{\text{Per}(f)}$ be the closure of the periodic points of $f$, $\text{NW}(f)$ be the set of nonwandering points for $f$, and $\text{CR}(f)$ be the set of chain recurrent points for $f$.
   (a) Define a nonwandering point
   (b) Prove that $\overline{\text{Per}(f)} \subset \text{NW}(f) \subset \text{CR}(f)$.
   (c) Show be example that $\overline{\text{Per}(f)}$ may not equal $\text{CR}(f)$.

(7) Let $f$ be a diffeomorphism and $p$ a periodic point for $f$. Let $H_p$ be the equivalence class of periodic points heteroclinically related to $p$, i.e. $q \sim p$ if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^s(q) \pitchfork W^u(p) \neq \emptyset$. Let $\Lambda_p$ be the closure of $H_p$. Prove that $\Lambda_p$ is transitive.

(8) Let $T : X \to X$ be a measure preserving transformation of the measure space $(X, \mathcal{B}, \mu)$.
   (a) Define what it means for $T$ to be ergodic.
   (b) Define what it means for $T$ to be mixing.
   (c) Prove that any mixing transformation is ergodic.