WINTER 2012 - PH.D. PRELIMINARY EXAMINATION ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

(1) Prove that the planar system

$$\dot{x} = x + y - x^3$$
$$\dot{y} = -x + y - y^3$$

has a nonconstant periodic solution.

(2) Find all real numbers a such that the initial-value problem

$$\ddot{x} = x + ae^{-t}$$
$$x(0) = 1$$
$$\dot{x}(0) = 0$$

has a solution $x(t)$ that is bounded as $t \to \infty$.

(3) Prove that for every $\varepsilon > 0$ there exists $(x_0, y_0) \in \mathbb{R}^2$ such that $0 < x_0^2 + y_0^2 < \varepsilon^2$ and the solution $(x(t), y(t))$ of the initial-value problem

$$\dot{x} = (x^2 + y^2)(4y - x^2)$$
$$\dot{y} = (x^2 + y^2)(x + 7y^3)$$
$$x(0) = x_0$$
$$y(0) = y_0$$

satisfies $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$.

(4) Consider the initial-value problem

$$\dot{x} = x\sin(x - t) + 1$$
$$x(\tau) = 0.$$

It is straightforward to check that $x(t) := t$ is the solution if $\tau = 0$ and that the solution x will be different for different initial times τ. Let's write the solution x as $x(t, \tau)$ to make explicit its dependence on τ. Compute $x_\tau(t, 0)$ (where the subscript τ stands for partial differentiation with respect to τ).

(5) Let $f : X \to X$ be a homeomorphism of a complete compact metric space. Suppose that $\bigcup_{n=-\infty}^{\infty} f^n U = X$ for every nonempty open set $U \subset X$. Prove that there is a point $x \in X$ such that the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is dense in X.

(6) Suppose that $f : M \to M$ is a diffeomorphism and $p \in M$ is a hyperbolic fixed point for f. Let $q \in W^s(p) \pitchfork W^u(p)$. Prove that $\Lambda = \{p\} \cup \mathcal{O}(q)$ is a hyperbolic set for f. (where $\mathcal{O}(q) = \{\bigcup_{n\in\mathbb{Z}} f^n(q)\}$)

(7) Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of a diffeomorphism $f : S^1 \to S^1$. Suppose that f has a periodic point. Prove (from first principles) that there is a rational number p/q such that

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \frac{p}{q}$$

for all $x \in \mathbb{R}$. Conversely, suppose that the above limit exists and is rational for some $x \in \mathbb{R}$. Prove that f has a periodic point.

(8) Assume that $f : [0,1] \to [0,1]$ is continuous and there exists two disjoint intervals I_1 and I_2 such that $f(I_1) \subset I_1 \cup I_2$ and $f(I_2) \supset I_1$. Show that there the topological entropy of f is at least $\ln(\frac{1+\sqrt{5}}{2})$.