JAN 2013- PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS

Instructions: Give solutions to the following problems and show all your work.

(1) Consider the system

$$x' = x^2 + y$$
$$y' = xy + a$$

where a is a parameter.
  (a) Find all equilibrium points and compute the linearized equation at each.
  (b) Describe the behavior of the linearized system at each equilibrium point.

(2) Let $A(t)$ be a continuous family of $n \times n$ matrices and let $P(t)$ be the matrix solution to the initial value problem $\frac{d}{dt}P = A(t)P, P(0) = P_0$. Show that

$$\det P(t) = (\det P_0)\exp\left(\int_0^t \mathrm{Tr}A(s)ds\right).$$

(3) A solution $x(t)$ of a system of differential equations is called recurrent if $x(t_n) \to x(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solutions.

(4) Let γ be a closed orbit for a planar system, and let U be the bounded, open region inside γ. Show that γ is not simultaneously the omega and alpha limit set of points of U. Use this fact and the Poincaré-Bendixson theorem to prove that U contains an equilibrium that is not a saddle.

(5) Let U be an open set of $\mathbb{R}^n$ containing x_0. Suppose that $f : U \to \mathbb{R}^n$ is C^1 and $f(x_0) = 0$. Suppose further that there is a C^1 function $V : U \to \mathbb{R}$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Prove
  (a) if $\dot{V}(x) = \mathrm{grad}V(x) \cdot f(x) \leq 0$ for all $x \in U$, then x_0 is stable.
  (b) if $\dot{V}(x) < 0$ for all $x \in U - \{x_0\}$, then x_0 is asymptotically stable.

(6) Consider the following differential equation

$$x' = Ax + f(x)$$

where $x \in \mathbb{R}^2$, A is a 2×2 matrix, and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is globally Lipschitz continuous and $f(0) = 0$. Assume that A has a positive and a negative eigenvalues. Prove that there exists a constant $\delta > 0$ such that if $Lip(f) < \delta$, then the global stable manifold $W^s(0)$ is given by the graph of a Lipschitz continuous function.