Instructions: Give solutions to 6 of the following 7 problems.

(1) Sketch the phase portraits of
\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= x - x^2 + \epsilon y
\end{align*}
\]
when $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$.

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. Prove the initial value problem
\[
\frac{dx}{dt} = \frac{1}{1 + |f(x)|^2}f(x), \quad x(0) = x_0 \in \mathbb{R}^n
\]
has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$.

(3) Suppose that $V$ is a smooth function defined on an open neighborhood $U$ of the rest point (equilibrium point) $x_0$ of the autonomous system $x' = f(x)$ such that $V(x_0) = 0$ and $\dot{V}(x) = \nabla V(x) \cdot f(x) > 0$ on $U \setminus \{x_0\}$. If $V$ has a positive value somewhere in each open set containing $x_0$. Prove $x_0$ is unstable.

(4) Suppose that $\gamma$ is a periodic orbit of a smooth flow defined on $\mathbb{R}^2$. Use Zorn’s lemma to prove that $\gamma$ surrounds a rest point of the flow.

(5) Consider the initial value problem
\[
x' = Ax + g(x, t), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n.
\]
where $A$ is a constant matrix with spectrum in the left half plane and $g(x, t)$ is smooth and there are constants $a > 0$ and $k > 0$ such that
\[
|g(x, t)| \leq k|x|^2
\]
for all $t$ and $|x| < a$. Prove that there are positive constants $C, b, \text{ and } \alpha$ such that the solution $x(t)$ of the above initial value problem satisfies
\[
|x(t)| \leq C|x_0|e^{-\alpha(t-t_0)}
\]
for $t > t_0$ and $|x_0| \leq b$.

(6) Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t, x)$ where $x \in \mathbb{R}^n$, $f$ and $h$ are $C^2$ functions, and $h(t, x)$ is $T$-periodic in $t$, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point $p^*$, then the perturbed equation $x' = f(x) + \epsilon h(t, x)$ has a unique periodic solution $p(t, \epsilon)$ such that
\[
p(t, \epsilon) - p^* = O(\epsilon).
\]

(7) Consider the following differential equation
\[
x' = Ax + f(x)
\]
where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \times \mathbb{R}^2$ is globally Lipschitz continuous and $f(0) = 0$. Assume that $A$ has a positive and a negative eigenvalues. Prove that there exists a constant $\delta > 0$ such that if $\text{Lip}(f) < \delta$, then the global unstable manifold $W^u(0)$ is given by the graph of a Lipschitz continuous function.