MAY 2013- PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUAITONS

Instructions: Give solutions to 6 of the following 7 problems.

(1) Sketch the phase portraits of
$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = x - x^2 + \epsilon y$$
when $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$.

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. Prove the initial value problem
$$\frac{dx}{dt} = \frac{1}{1 + |f(x)|^2} f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$
has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$.

(3) Suppose that V is a smooth function defined on an open neighborhood U of the rest point (equilibrium point) x_0 of the autonomous system $x' = f(x)$ such that $V(x_0) = 0$ and $\dot{V}(x) = grad(V(x)) \cdot f(x) > 0$ on $U \setminus \{x_0\}$. If V has a positive value somewhere in each open set containing x_0. Prove x_0 is unstable.

(4) Suppose that γ is a periodic orbit of a smooth flow defined on $\mathbb{R}^2$. Use Zorn's lemma to prove that γ surrounds a rest point of the flow.

(5) Consider the initial value problem
$$x' = Ax + g(x,t), \quad x(t_0) = x_0, \ x \in \mathbb{R}^n.$$
where A is a constant matrix with spectrum in the left half plane and $g(x,t)$ is smooth and there are constants $a > 0$ and $k > 0$ such that
$$|g(x,t)| \leq k|x|^2$$
for all t and $|x| < a$. Prove that there are positive constants C, b, and α such that the solution $x(t)$ of the above initial value problem satisfies
$$|x(t)| \leq C|x_0|e^{-\alpha(t-t_0)}$$
for $t > t_0$ and $|x_0| \leq b$.

(6) Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t,x)$ where $x \in \mathbb{R}^n$, f and h are C^2 functions, and $h(t,x)$ is $T-$periodic in t, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point p^*, then the perturbed equation $x' = f(x) + \epsilon h(t,x)$ has a unique periodic solution $p(t,\epsilon)$ such that
$$p(t,\epsilon) - p^* = O(\epsilon).$$

(7) Consider the following differential equation
$$x' = Ax + f(x)$$
where $\in x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \times \mathbb{R}^2$ is globally Lipschitz continuous and $f(0) = 0$. Assume that A has a positive and a negative eigenvalues. Prove that there exists a constant $\delta > 0$ such that if $Lip(f) < \delta$, then the global unstable manifold $W^u(0)$ is given by the graph of a Lipschitz continuous function.