QUALIFYING EXAM –Differential Equations, May 2010

Name:

Choose six problems

1. Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the system

$$x' = g(x), \quad y' = f(x)y,$$

has at most one solution on any interval , for a given initial value.

2. Consider the differential equation $\dot{x} = f(x)$ on the plane. Suppose that there are two nonconstant periodic solutions for which one surrounds another and the region bounded by these two periodic solutions contains neither a periodic solution nor an equilibrium. Prove that these two periodic solutions cannot both be orbitally stable.

3. Show that

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + e^{-|x|}\cos t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

has a 2π-periodic solution.

4. Consider the following differential equation

$$\dot{x} = Ax + f(x),$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and $f(0) = 0, f'(0) = 0$. Assume that A has a positive and a negative eigenvalues. Prove that $x = 0$ is unstable.

5. Determine the stability of equilibrium $(x, y, z) = 0$ of the following equations

$$\begin{aligned} x' &= 2y(z-1), \\ y' &= -x(z-1), \\ z' &= -z^3. \end{aligned}$$

6. Let $x(t, x_0)$ be the solution of $x' = f(x)$ with initial condition $x(0, x_0) = x_0$, where $x \in \mathbb{R}^n$ and f is a Lipschitz continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose $\{x(t, x_0), t \geq 0\}$ is bounded. Prove that the omega-limit set $\omega(x_0)$ is nonempty, compact, invariant and connetcted.

7. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Let $z \in \omega(x)$, the omega-limit set of x of the gradient system $x' = -\text{grad}V(x)$. Prove that z is an equilibrium.