Ph.D. Qualifying Exam: Applied Math/ODE
May 2017

Instructions: answer 6 of the 8 questions. If you answer more than 6 questions, the first 6 questions will be graded.

1. Consider the system

$$\dot{x} = x^2 - y$$
$$\dot{y} = x - \epsilon$$

where ϵ is a real parameter. Find the fixed point(s) of the system and compute the linearized system of equations about each of these point(s). Then determine the behavior of the linearized system at each of these point(s) for $\epsilon \in \mathbb{R}$.

2. Consider the system

$$\dot{x} = -y + x(r^4 - 3r^2 + 1)$$
$$\dot{y} = x + y(r^4 - 3r^2 + 1)$$

where $r^2 = x^2 + y^2$. Show that system has a periodic orbit in the annular region $A = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 : \ 1 < |\mathbf{x}| < 2\}$.

3. Prove that the zero solution of the system

$$\dot{x}_1 = -x_2 + 2x_2x_3$$
$$\dot{x}_2 = x_1 - x_1x_3$$
$$\dot{x}_3 = x_2^2x_3 + x_1x_2$$

is stable.

4. Consider the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n.$$

Show that if all eigenvalues of A have negative real parts, then

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$$

using the Jordan cannonical form of A.

5. For a compact metric space X, suppose $f : X \to X$ is continuous and that a nonempty subset Y of X is invariant for f. Prove that if f is topologically mixing on Y, then f is topologically transitive on Y.

6. Suppose the continuous functions $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, where X and Y are compact metric spaces. Prove that if f has sensitive dependence on initial conditions, then g has sensitive dependence on initial conditions.

7. Assume that $f : [0,1] \to [0,1]$ is continuous and there are two disjoint compact subintervals I_1 and I_2 of $[0,1]$ such that $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1$. Prove that the topological entropy of f is at least $\ln((1+\sqrt{5})/2)$.

8. For a smooth manifold M, suppose that $f : M \to M$ is a diffeomorphism and $p \in M$ is a hyperbolic fixed point for f. Suppose $W^s(p)$ and $W^u(p)$ intersect transversally at a point $q \in M$. Prove that $\Lambda = \{p\} \cup \{f^n(q) : n \in \mathbb{Z}\}$ is a hyperbolic set for f.