SPRING 2015 - PH.D. PRELIMINARY EXAMINATION ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first 6 that appear.

(1) State and prove either the Cauchy-Peano Existence Theorem or the Picard-Lindelöf Existence and Uniqueness Theorem for initial value problems. You may assume the Arzelá-Ascoli Theorem and the Contraction Mapping Principle.

(2) Consider a linear system $X' = AX$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(a) Find the matrix exponential e^{At}.
(b) Find all X_0 such that solutions $X(t)$ satisfying $X(0) = X_0$ are unbounded for $t \geq 0$.
(c) Find all X_0 such that solutions $X(t)$ satisfying $X(0) = X_0$ are unbounded for $t \leq 0$.

(3) Let p be a fixed point of a C^1 system $x' = f(x)$ for $x \in \mathbb{R}^n$.
(a) Give the definition of *positive stability* and *positive asymptotic stability* of p (Here positive means $t \geq 0$). Then give similar definitions for *negative stability* and *negative asymptotic stability* for $t \leq 0$.
(b) Prove that p cannot be simultaneously positively asymptotically stable and negatively stable.
(c) Give an example of a system with an isolated fixed point that is simultaneously positively stable and negatively stable.

(4) Consider a system

$$\begin{aligned} x' &= x - rx - ry + xy \\ y' &= y - ry + rx - x^2 \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$.
(a) Find the ω-limit set $\omega(p)$ for every $p \in (x, y) \in \mathbb{R}^2$.

(b) Sketch the phase portrait.

(5) Answer the following:
   (a) Define topological conjugacy.
   (b) State the Harman-Grobman Theorem for maps.
   (c) Find a topological conjugacy between the following two systems defined on $\mathbb{R}^2$

$$f(x,y) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } g(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(6) Let M be a compact connected smooth manifold without boundary and $f : M \to M$ a diffeomorphism. Prove that if $\Lambda \subset M$ is a locally maximal hyperbolic set such that $f : \Lambda \to \Lambda$ is transitive and Λ has nonempty interior, then $\Lambda = M$.

(7) Let X be a complete metric space with a countable basis and $f : X \to X$ a continuous map. Assume that for every open set U of X the set $\bigcup_{n \leq 0} f^n(U)$ is dense in X. Prove that there is a point in X whose forward orbit is dense.

(8) Let M be a compact connected smooth manifold without boundary and $f : M \to M$ be a diffeomorphism. Let $p_1, ..., p_k$ be hyperbolic periodic points (of possibly different periods) for f. Suppose that $W^u(p_i)$ intersects $W^s(p_{i+1})$ transversally at points q_i for $1 \leq i \leq k$ where $p_{k+1} = p_1$. The points q_i are called transverse heteroclinic points. Prove that any neighborhood of $\{\mathcal{O}(p_i) \cup \mathcal{O}(q_i)\}_{i=1}^k$ contains a horseshoe (that is the neighborhood contains a set Λ such that for some $k > 0$ the map f^k restricted to Λ is hyperbolic and topologically conjugate to the full 2-shift).