MAY 2011 PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS

Instructions: Give solutions to exactly 6 of the following 7 problems.

(1) Prove that the origin is an uniformly asymptotically stable solution for the equation

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -x - y^3(1 - 5\sin t).$$

(2) Let $A(t)$ be a continuous $n \times n$ matrix with period T. Suppose that

$$x' = A(t)x$$

has no periodic solutions of period T and there is no non-zero constant solutions. Let $g(t) \in \mathbb{R}^n$ be any continuous vector function with period T. Prove

$$x' = A(t)x + g(t)$$

has a periodic (possibly constant) solution of period T.

(3) Use the Poincare-Bendixson Theorem to prove the Brouwer fixed point theorem for a C^1 mapping.

(4) Let A be a 2×2 matrix with one positive eigenvalue and one negative eigenvalue and $g(t)$ be a bounded continuous function from $(-\infty, \infty)$ to $\mathbb{R}^2$. Prove that

$$x' = Ax + g(t)$$

has an unique bounded solution over $(-\infty, \infty)$.

(5) Prove that

$$x' = -3x + \frac{1}{1 + |x|^4} \sin t \cos t$$

has 2π periodic solution.

(6) Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t, x)$ where $x \in \mathbb{R}^n$, f and h are C^2 functions, and $h(t, x)$ is $T-$periodic in t, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point p^*, then the perturbed equation $x' = f(x) + \epsilon h(t, x)$ has a unique periodic solution $p(t, \epsilon)$ such that

$$p(t, \epsilon) - p^* = O(\epsilon).$$

(7) Consider the following differential equation

$$x' = Ax + f(x)$$

where $\in x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \times \mathbb{R}^2$ is globally Lipschitz continuous and $f(0) = 0$. Assume that A has a positive and a negative eigenvalues. Prove that there exists a constant $\delta > 0$ such that if $Lip(f) < \delta$, then the global unstable manifold $W^u(0)$ is given by the graph of a Lipschitz continuous function.