FALL 2015 TOPOLOGY QUALIFYING EXAM

Rules: Show all your work. If you think you've found a mistake in a problem or you've found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do 7 out of 8 problems.

(1) Prove that a nonempty compact metric space in which every point is an accumulation point contains uncountably many points.

(2) Prove that if X and Y are path connected spaces with basepoints x_0, y_0 respectively then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(3) Let E be a connected covering space of the connected, locally path connected space X with covering projection p. Let $g : E \to E$ be a deck transformation, that is, $p \circ g = p$. Show that either g has no fixed points or that every point of E is a fixed point of g.

(4) Give an example of a connected finite complex X and a connected finite-sheeted covering space E of X with the property that the second homology of E is not isomorphic to the second homology of X. Give a complete proof, clearly stating all standard theorems which you apply.

(5) Show that the Euler characteristic of a compact orientable manifold of odd dimension is 0.

(6) Recall that $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} | \, \text{determinant}(A) = 1\}$. Show that $\mathrm{SL}(n, \mathbb{R})$ is a smoothly embedded submanifold of $\mathbb{R}^{n^2}$ and compute its dimension.

(7) Prove that there does not exist any immersion $f : S^2 \to \mathbb{R}^2$.

(8) Give (with proof) an example of a closed but not exact 2-form on $\mathbb{R}^3 - \{0\}$.