Instructions: Do all seven problems. Do not make assumptions or use known theorems which trivialize a problem.

(1) Let $X$, $Y$, and $Z$ be a path connected and locally path connected, with $q: X \to Y$ and $r: Y \to Z$ continuous.

(a) Show that if $r$ and $r \circ q$ are covering maps, then so is $q$.
(b) Show that if $q$ and $r \circ q$ are covering maps, then so is $r$.

(2) Let $X$ be a linearly ordered set with the order topology. Show that if $X$ has the supremum property (every nonempty set which is bounded above has a supremum), then closed intervals of $X$ are compact.

(3) Let $E$ be a fiber bundle over $B$ with fiber $F$. That is, there is a continuous surjection $p: E \to B$ with the property that for any $e \in E$ there exists a neighborhood $U$ of $p(e)$ in $B$ and a homeomorphism $h: p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$
P^{-1}(U) \xrightarrow{h} U \times F \\
p \downarrow \quad \downarrow \text{projection}
$$

Prove that if $F$ is path connected and $p(e_0) = b_0$, then $P_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is onto.

(4) Consider the submanifold $M$ of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = 1$.

(a) Show that the vector field $X = \frac{xz}{1 + z^2} \frac{\partial}{\partial x} + \frac{yz}{1 + z^2} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ is tangent to $M$.

That is, there is a vector field $Y$ on $M$ such that $i_*(Y(m)) = X(m)$, $\forall m \in M$.

(b) Show that the two form $w = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$ restricts to a nonvanishing form on $M$. (Hint: cylindrical coordinates.)

(c) Show that the flow of $Y$ on $M$ preserves $i^*(w)$.

(5) Let $A$ be the union of two once linked embedded circles in $S^3$. Let $B$ be the union of two unlinked circles in $S^3$. Show that the cohomology groups of $S^3 - A$ and $S^3 - B$ are isomorphic, but the cohomology rings are not.

(6) Let $M$ and $N$ be closed $n$-manifolds and $P$ be the connected sum of $M$ and $N$. Show $\chi(P) = \chi(M) + \chi(N) - 2$ if $n$ is even and $\chi(P) = \chi(M) + \chi(N)$ if $n$ is odd ($\chi =$ Euler characteristic).

(7) (a) Suppose $\omega$ is a smooth exact $k$-form. Show that $\omega \wedge \omega$ is an exact $2k$-form.

(b) Suppose $\omega$ is a smooth 2-form on $S^4$. Show that $\omega \wedge \omega$ vanishes somewhere.