Topology Qualifying Exam, May 2012

Instructions: Do all problems. **Policy on misprints:** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

1. Let $M_1, M_2, \ldots, M_k$ be smooth manifolds.

(a) It can be shown that $M_1 \times M_2 \times \cdots \times M_k$ is second countable and Hausdorff. Assuming this, show that $M_1 \times M_2 \times \cdots \times M_k$ is a smooth manifold.

(b) Let $\pi_j\colon\ M_1 \times \cdots \times M_k \to M_j$ denote projection to M_k. Let $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$. Show that the map $\alpha\colon\ T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ defined by

$$\alpha(X) = ((\pi_1)_*(X), \ldots, (\pi_k)_*(X))$$

is a linear isomorphism.

2. Let M be a smooth n–manifold with boundary. Show ∂M is a topological $(n-1)$–manifold with a unique smooth structure such that the inclusion $\iota\colon\ \partial M \to M$ is a smooth embedding.

3. Let ω be the 2–form on $\mathbb{R}^3$ given by

$$\omega = z\, dx \wedge dy.$$

(a) Let $F\colon\ [0, \pi] \times [0, 2\pi] \to S^2$ be the spherical coordinate parameterization:

$$F(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

Compute $F^*(\omega|_{S^2})$.

(b) Evaluate $\int_{S^2} \omega$.

4. Find a presentation for the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle in one torus $S^1 \times \{x_0\}$ with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

5. Let X be a topological space. Recall that the suspension of X, denoted SX, is defined to be the quotient of $X \times [0, 1]$ obtained by collapsing $X \times \{0\}$ to one point, and $X \times \{1\}$ to another point.

Show that there are isomorphisms $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n.

6. Let M be a compact simply connected 4–manifold with $\operatorname{rank} H_2(M) = 2$. Find $H^k(M; \mathbb{Z})$ and $H_k(M; \mathbb{Z})$ for all k. State all theorems you use.