Instructions: Show all your work. If you think you’ve found a mistake in a problem or you’ve found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do 7 out of 8 problems.

(1) Let $X$ be a space obtained by gluing all 3 sides of a solid (2-dimensional) triangle together to one edge via homeomorphisms sending vertices to vertices. Show that there are two such possibilities for $X$ up to homeomorphism. Compute the corresponding fundamental groups. Compute all of the covering spaces of such spaces.

(2) Suppose $X$ is a space which is covered by a simply connected, locally path connected, compact space. Suppose $G$ is a torsion-free group and $Y$ is a $K(G, 1)$ space (i.e., $Y$ has a universal cover which is contractible and $\pi_1(Y) = G$.) Show that any continuous map from $X$ to $Y$ is contractible (homotopic to a constant map). Partial credit will be given for completely working out the case where $X = \mathbb{R}P^2$ and $Y = S^1 \times S^1 \times S^1$.

(3) For all $k$ compute $H_k(Z, \partial Z)$ where $Z = D^2 \times S^1$.

(4) Show that if $S^n$ admits a nowhere vanishing vector field, then $n$ is odd.

(5) Let $S$ be a “wild” embedded 2-sphere in $S^3$ (i.e., $\pi_1(S^3 - S, x_0)$ is non-trivial for some choice of $x_0$). Compute $H_1(S^3 - S)$.

(6) Suppose that $X$ is a simply connected, compact, 5-dimensional manifold such that $H^3(X)$ is finite. Show that $H^2(X)$ is finite.

(7) Give (with proof) an example of a closed but not exact 1-form on $\mathbb{R}^2 - \{0\}$.

(8) Show that $\text{SL}(n, \mathbb{R})$ is a smoothly embedded submanifold of $\mathbb{R}^{n^2}$ and compute its dimension.