Topology Qualifying Exam Spring 2015

Instructions: Show all your work. If you think you've found a mistake in a problem or you've found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do five problems.

(1) Show that a compact Hausdorf space is normal.

(2) Let S_g be the compact orientable surface of genus g. Consider the 4-manifold $M = S_2 \times S_5$. Find two non-homeomorphic two-sheeted covering spaces for M. Justify your answer.

(3) (a) Let X and Y be topological spaces with X compact. Let $\mathcal{G}$ be a collection of basic open sets in $X \times Y$ that forms an open cover of $X \times Y$. Show that if $\mathcal{G}$ doesn't have a finite subcover then there is a point $x \in X$ such that $\{x\} \times Y$ is not covered by a finite subset of $\mathcal{G}$.

(b) State the Tychonoff theorem and prove it using transfinite induction.

(4) Let M be a compact manifold without boundary of dimension n. Suppose that $f : M \to \mathbb{R}^n$ is smooth. Prove that f is not everywhere non-singular.

(5) Let M be a compact orientable n-manifold with boundary, such that the boundary ∂M is a rational homology sphere (that is $H_*(\partial M, \mathbb{Q}) \cong H_*(S^{n-1}, \mathbb{Q})$). Show that if n is odd then

$$\chi(M) = 1.$$

(6) First calculate the cohomology ring of $\mathbb{CP}^2$. Then calculate the cohomology ring of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

(7) Show that every smooth manifold M admits a Riemannian structure. That is it admits an inner product $\langle\, ,\, \rangle_m$ on M_m for $m \in M$, such that if X, Y are smooth vector fields on M then $\langle X, Y \rangle_m$ is a smooth function on M.