(1) Construct a 4-fold covering space \((X, x)\) of the following graph \((Y, y)\) with the property that \(a^2 \in \text{Im}(\pi_1(X, x) \hookrightarrow \pi_1(Y, y))\), but \(a \not\in \text{Im}(\pi_1(X, x) \hookrightarrow \pi_1(Y, y))\).

(2) Let \(X\) be a compact metric space and \(f : X \to X\) an isometric embedding. Show \(f\) is onto.

(3) Let \(X\) be a Hausdorff space and \(\{f_\alpha\}_{\alpha \in J}\) a family of continuous functions \(f_\alpha : X \to \mathbb{R}\) satisfying the requirement that \(\forall\ x \in X\) and \(\forall\ U\) neighborhood of \(x\), there exists \(\alpha \in J\) with \(f_\alpha(x) > 0\) and \(f(y) = 0\ \forall\ y \not\in U\). Prove that \(F : X \to \mathbb{R}^J\) defined by \(F(x) = (f_\alpha(x))_{\alpha \in J}\) is an embedding.

(4) Show that \(P^2 \# P^2 \# P^2 \cong T \# P^2\) where \(T\) is the torus and \(P^2\) the projective plane.

(5) Let \(M\) be an \(n\)-dimensional smooth closed compact manifold, and let \(TM\) be the total space of its tangent bundle. If \(H^k_{dR}(TM)\) denotes the \(k\)th de Rham cohomology space of \(TM\), prove that
\[
H^k_{dR}(TM) \cong 0
\]
for all \(k > n\).

(6) Prove that for any smooth compact manifold \(M\) of dimension \(n \geq 1\), there exists a smooth map \(f : M \to S^n\) of degree 1.

(7) Prove that there does not exist any immersion \(f : S^2 \to \mathbb{R}^2\).

(8) Consider the vector fields
\[
V = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{and} \quad W = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}
\]
on \(\mathbb{R}^3\), where \(a, b,\) and \(c\) are real constants. Let \(\varphi_s\) and \(\psi_t\) denote the global flows of \(V\) and \(W\) respectively.

(a) What geometric transformations do \(\varphi_s\) and \(\psi_t\) correspond to?

(b) For what values of \(a, b,\) and \(c\) do \(\varphi_s\) and \(\psi_t\) commute, i.e. \(\varphi_s \circ \psi_t = \psi_t \circ \varphi_s\) for all \(s\) and \(t\)?
(9) Let $A$ be the subset of $\mathbb{R}^3$ consisting of the spheres of radius 1 and 2 about 0 together with a circle of radius 2 centered at $(0, 0, 2)$ passing through the origin. Find the homology groups of $A$ with $\mathbb{Z}$ coefficients in simplest form.

(10) Let $M$ be a compact orientable 7-dimensional manifold without boundary. Suppose $a \in H_2(M) a \neq 0$, but $8a = 0$. Show that there is $b \in H_y(M) b \neq 0$, but $8b = 0$.

(11) Let $X$ be a finite simplicial complex. Find $H^*(X, \mathbb{Z}_8)$ and $H_*(X, \mathbb{Z}_8)$ given that

\[
H_P(X, \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } p = 0, 4 \\
\mathbb{Z}_2 & \text{if } p = 1, 5 \\
\mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\
\mathbb{Z}_{18} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z} & \text{if } p = 3 \\
0 & \text{if } p > 5
\end{cases}
\]

(12) Let $A$ and $B$ be disjoint embeddings of $S^1$ into $\mathbb{R}^3$. Find $H_*(\mathbb{R}^3 - (A \cup B))$. 