1. Prove that every perfect subset of a complete metric space is uncountable.

2. Prove that $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX)$ where $SX$ is the suspension of $X$, i.e. $SX = X \times [0, 1]/\sim$ where $(x, t) \sim (y, t)$ for all $x, y \in X$ and $t \in \{0, 1\}$.

3. Let $X$ be a path-connected, locally path-connected topological space with finite fundamental group and $\mathbb{S}^k$ the unit sphere in $\mathbb{R}^{k+1}$.
   
   (a) Prove that any continuous map $f : X \to \mathbb{S}^1 \times \mathbb{S}^1$ is homotopic to a constant map.
   
   (b) Show that the quotient map $q : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ obtained by collapsing $\mathbb{S}^1 \times \{x_0\} \vee \{x_0\} \times \mathbb{S}^1$ to a point for $x_0 \in \mathbb{S}^1$ is not nullhomotopic.

4. Let $X_i$ be a CW complex for $i \in \{1, \ldots, n\}$ and $\mathbb{S}^k$ the unit sphere in $\mathbb{R}^{k+1}$.
   
   (a) Prove that $\tilde{H}_*(\bigvee_{i=1}^n X_i) = \bigoplus_{i=1}^n \tilde{H}_*(X_i)$.
   
   (b) Compute $H_*(\mathbb{S}^1 \vee \mathbb{S}^2)$.

5. Let $M_n$ be the space of $n \times n$ real matrices, with topology obtained through the natural identification of $M_n$ with $\mathbb{R}^{n^2}$. Let $S_n \subset M_n$ be the subspace of symmetric matrices, and let $F : M_n \to S_n$ be the smooth map given by $F(A) = AA^T$.
   
   (a) Show that map induced by $F$ on the tangent space $\mathcal{L}_A M_n \to \mathcal{L}_{F(A)} S_n$
   
   is given by $\mathcal{L}_A F(B) = AB^T + BA^T$. Note that here we are using the identification $\mathcal{L}_A M_n \cong \mathbb{R}^{n^2} \cong M_n(\mathbb{R})$.
   
   (b) Show that the $n \times n$ identity matrix $\text{Id} \in S_n$ is a regular value of the map $F$.
   
   (c) Show that $O(n) = F^{-1}(\text{Id})$ is a smooth submanifold of $M_n$, and determine its dimension.
   
   (d) Determine the tangent space of $O(n)$ at $\text{Id} \in O(n)$. (Describe your answer as a familiar subspace of $M_n$.)

6. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that the complement $\mathbb{R}^n \setminus M$ is both connected and simply-connected.

7. Define the deRham cohomology groups $H^i_{dR}(M)$ of a smooth manifold $M$. Use this definition to directly compute $H^i_{dR}(S^1)$ for $i = 0, 1, 2, \ldots$, where $S^1 = \mathbb{R}/\mathbb{Z}$.

8. Let $M$ be a smooth compact 3-manifold with boundary $\partial M \neq \emptyset$. If $\theta_1$ and $\theta_2$ are two 1-forms on $\partial M$ so that $\theta_1 \wedge \theta_2$ is a volume form on $\partial M$, show that $\theta_1$ and $\theta_2$ cannot both extend to closed forms on $M$. Can you find such $M$, $\theta_1$, and $\theta_2$ so that $\theta_1$ extends to a closed 1-form on $M$?