TOPOLOGY QUALIFYING EXAM FALL 2010

Instructions: Do all problems. Each problem is worth 10 points. For the purposes of this test, the circle $S^1 = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$ and the closed 2-disk $D^2 = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq 1\}$ are given the subspace topology inherited from the plane $\mathbb{R}^2$. Similarly S^2 is defined to be the subspace of unit vectors in $\mathbb{R}^3$. Finally, the real projective plane is defined to be the quotient space

$$\mathbb{R}P^2 = \frac{S^2}{s = -s, \ \forall\, s \in S^2}.$$

You may use the results from any problem or part of a problem even if you do not correctly answer that part.

(1) Show that a topological space is locally path connected if and only if every component of each of its open sets is itself open.

(2) (a) Prove that every continuous map from the closed 2-disk, D^2 to itself must have a fixed point.
  (b) Show that if $f : S^2 \to S^2$ is a continuous map and $f(x) \neq f(-x) \forall x \in S^2$ then f is surjective.

(3) Calculate the fundamental group of S^1 directly from the definitions.

(4) If S is a compact orientable surface of genus g and $\tilde{S}$ is a finite covering space of S of degree r, show that $\tilde{S}$ is a surface and calculate its genus.

(5) Use the Seifert-van Kampen theorem to calculate the fundamental group of the Klein bottle.

(6) Let $T^2 = S^1 \times S^1$ and let $f : T^2 \to T^2$ be defined by $f(a,b) = (-b,a)$. The corresponding *mapping torus* is defined by

$$M = \frac{T^2 \times [0,1]}{(a,0) = (f(a),1), \ \forall\, a \in T^2}$$

with the induced quotient topology.
  (a) Show that $T^3 = S^1 \times S^1 \times S^1$ is a 4-fold covering space for M.
  (b) Show that M is a 3-manifold and that $\pi_1(M)$ contains $\mathbb{Z}^3$ as an index 4 subgroup.
  (c) Use the Mayer-Vietoris theorem to calculate $H_i(M, \mathbb{Z})$ for all i.

(7) Show that the Euler characteristic of any 3-dimensional compact orientable manifold is 0. If you wish, you may use the fact that the Euler characteristic of any finite l-dimensional simplicial complex K can be calculated as $\chi(K) = \sum_{i=0}^{l} (-1)^i b_i$, where the i-th Betti number b_i is defined to be the $\mathbb{Z}$-rank (the rank of the torsion-free part) of the free Abelian group $H_i(K, \mathbb{Z})$. If you choose to use the Universal Coefficients theorem or the Poincaré Duality theorem, then give a complete statement of them.

(8) Let $p \in \mathbb{R}P^2, s \in S^2$ and $c \in S^1$. Let $Z = \mathbb{R}P^2 \vee S^2 \vee S^1$. Recall that this defines Z as the quotient space given by

$$Z = \frac{\mathbb{R}P^2 \coprod S^2 \coprod S^1}{p = s = c},$$

where $\coprod$ denotes a *disjoint union*. Calculate all of the integral homology groups and rational cohomology groups of Z, that is, calculate $H_i(Z, \mathbb{Z})$ and $H^i(Z, \mathbb{Q})$ for all i.