Algebra Ph.D. Qualifying Exam, August 2013
Answer all questions. Partial credit will be given.

1. Prove Cauchy's theorem, that if G is a finite group and p is a prime dividing $|G|$, then G has an element of order p.

2. State and prove Eisenstein's irreducibility criterion for integral domains. (If you can't do it for arbitrary integral domains, do it for $\mathbb{Z}$ for partial credit.)

3. Let K/F be a finite extension of fields. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$.

4. Let M be a left R-module, for some ring R. Prove that if M is Noetherian, then every submodule of M is finitely generated. Also, give an example of a ring R and a finitely generated R-module M which is not Noetherian.

5. Let $p < q < r$ be primes. If G is a finite group of order pqr, prove that G has a normal Sylow subgroup for either p, q, or r.

6. Let R be a Unique Factorization Domain. Prove that any nonzero irreducible element is prime. (The converse is true in any integral domain.)

7. Prove that abelian groups are nilpotent, and nilpotent groups are solvable. Recall that for a group G we set $Z_0(G) = 1$, and we let $Z_1(G) = Z(G)$ be the center of G. We recursively define $Z_{i+1}(G)$ to be the subgroup of G containing $Z_i(G)$ such that
$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$
A group G is *nilpotent* if $Z_n(G) = G$ for some finite $n \geq 0$.

8. How many conjugacy classes of 5×5 matrices satisfy $x^5 - 1$ over $\mathbb{F}_{19}$? Justify your answer.

9. Let R be a ring, and assume that all idempotents $e^2 = e \in R$ are central. Prove that if $uv = 1$ for some $u, v \in R$ then $vu = 1$.

10. Determine the Galois group of $x^3 - 2x + 4$ over $\mathbb{Q}$. Do the same over $\mathbb{F}_2$ and $\mathbb{F}_3$.