PH.D. QUALIFYING EXAM SPRING 2013 - ALGEBRA

All rings have unity. Answer all questions.

1. Let G be a group of order 56 with a non-normal Sylow 7-subgroup.
   (a) Prove that G has a unique Sylow 2-subgroup S.
   (b) Prove that S is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. You may use the classification of groups of order 8 without proof. (Hint: Show that all non-identity elements of S are in the same orbit under the conjugation action of G on S.)
2. Prove or disprove: The ring of polynomials $\mathbb{C}[x, y]$ is a Euclidean domain.
3. Factor $x^5 + 5x + 5$ into irreducible factors in
   (a) $\mathbb{Q}[x]$;
   (b) $\mathbb{F}_2[x]$.
4. Let $\mathbb{F}_{81}$ be a field with 81 elements. Does the polynomial $x^3 + 2x + 1$ have a root in this field? Justify your answer.
5. Let ζ be a primitive 7th root of unity in $\mathbb{C}$. Determine the degree over $\mathbb{Q}$ of each of the following elements. Justify your answers.
   (a) $\zeta + \zeta^5$;
   (b) $\zeta^3 + \zeta^4$;
   (c) $\zeta^3 + \zeta^5 + \zeta^6$.
6. Prove that every group of order $992 (= 32 \times 31)$ is solvable.
7. List all abelian groups of order 1800, up to isomorphism. Each isomorphism class should appear on your list exactly once. Describe each group in both invariant factor and elementary divisor form, and indicate which is which.
8. Suppose A is an $n \times n$ matrix with entries in $\mathbb{Q}$. Suppose $A^p = I$, where p is a prime, and $A \neq I$. Prove that $n \geq p - 1$.
9. Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.
10. Recall that a ring R is simple if $R \neq 0$ and the only two-sided ideals of R are (0) and R. Prove that the center of a simple ring is a field.