FALL 2013 TOPOLOGY QUALIFYING EXAM

PROFS. CONNER AND PURCELL

Rules: Show all your work. If you think you’ve found a mistake in a problem or you’ve found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do 7 out of 8 problems.

(1) A map is monotone if the preimage of each point in the codomain is connected. Show that if $X$ is a simply connected CW-complex then no continuous monotone map from $X$ to $S^1$ is surjective by applying the lifting criterion to the universal cover of $S^1$.

(2) Compute the fundamental groups of: (a) the wedge of two projective planes and (b) the mapping torus over the antipodal map of $S^1$. Are these fundamental groups isomorphic?

(3) The $i$-th Betti number of a space $X$ is the $\mathbb{Z}$-rank of $H_i(X, \mathbb{Z})$ and is denoted $\beta_i(X)$. Let $M$ be a finite simplicial complex which is an orientable 4-manifold whose fundamental group is finite of order $n$ and let $\widetilde{M}$ be the universal covering space of $M$. Use two formulations of the Euler characteristic to show that $\beta_2(\widetilde{M}) = n \cdot \beta_2(M) + 2n - 2$.

(4) Let $Z$ be a genus 2 handlebody (that is, a “solid” two-holed surface or, equivalently, a regular neighborhood of a figure 8 in $\mathbb{R}^3$). For all $k$ compute $H_k(Z), H_k(\partial Z), H_k(Z, \partial Z)$ and $H^k(Z), H^k(\partial Z), H^k(Z, \partial Z)$.

(5) Let $F$ be a figure 8, a wedge of 2 circles, embedded in $S^3$. Note that $F$ is possibly knotted, that is, not isotopic to the standard figure 8. Compute $H_1(S^3 - F)$.

(6) Suppose $M$ is a smooth $n$-manifold, and the smooth map $f : M \to \mathbb{R}^{2n+1}$ is injective. Let $B$ be a closed subset of $M$, $U$ an open subset containing $B$, and $\phi_B$ a smooth bump function for $B$ supported in $U$. For any $b \in \mathbb{R}^{2n+1}$, define $g_b(x) : M \to \mathbb{R}^{2n+1}$ by

$$g_b(x) = f(x) + b \phi_B(x).$$

Show that there is some $b \neq 0$ in $\mathbb{R}^{2n+1}$ so that $g_b$ is injective.

(7) Let $\omega$ be an exact 2-form on $M = S^3 \times S^5$ and let $g : S^1 \times S^1 \to M$ be a smooth map. Prove that $\int_{S^1 \times S^1} g^* \omega = 0$.

(8) Recall that the set of $2 \times 2$ real matrices, $\mathbb{R}^{2 \times 2}$, is a smooth manifold diffeomorphic to $\mathbb{R}^4$. We see that $\mathbb{R}^{2 \times 2}$ can be naturally partitioned into 3 subsets: the matrices of rank 2, the matrices of rank 1, and the matrices of rank 0. Show that each of the partition elements is a smooth submanifold of $\mathbb{R}^{2 \times 2}$ and determine its dimension.