Math 113
Exam 1
Sept 26-27, 2016 (Late Day: Sept 28, 2016)

Encode your BYU ID in the grid below.

Instructions

1. Do not write on the barcode area at the top of each page, or near the four circles on each page.

2. Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely. Multiple choice questions are 5 points each.

3. For questions which require a written answer, show all your work in the space provided and justify your answer.

4. Simplify your answers.

5. No books, notes, or calculators of any type are allowed.

6. There is no time limit on this exam.
Part I: Multiple Choice Questions: Mark the correct answer. (5 points each)

1. Evaluate the integral \( \int \sin^4 x \cos^5 x \, dx \).
   \[ = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx \]
   \[ = \int \sin^4 y (1 - \sin^2 y)^2 \cos y \, dy \]
   \[ = \int u^4 (1-u^2)^2 \, du \]
   \[ = u^4 (1-2u^2+u^4) \]
   \[ = u^4 - 2u^6 + u^8 \]
   \[ = \frac{1}{5} x^5 - \frac{2}{7} x^7 + \frac{1}{7} x^9 \]
   (Sub for \( u(x) \))

2. What is the average of the function \( f(x) = x^2 \) on the interval \([0, 2]\)?
   \[ \text{avg \ value formula:} \quad \frac{1}{b-a} \int_a^b f(x) \, dx \]
   \[ \frac{1}{2-0} \int_0^2 x^2 = \frac{1}{2} \left[ \frac{1}{3} x^3 \right]_0^2 \]
   \[ = \frac{1}{2} \left[ \frac{1}{3} \cdot 2^3 \right] = \frac{8}{6} = \frac{4}{3} \]
3. Which of the following integrals accurately represents the volume $V$ obtained by rotating the region between $y = x^3$, $y = 1$, and $x = -1$, about the line $x = 2$ using the cylindrical shell method?

- $V = \int_{-1}^{1} 2\pi (2-x)(1-x^3)\,dx$
- $V = \int_{-1}^{1} \pi (2-x)^2(1-x^3)\,dx$
- $V = \int_{-1}^{1} 2\pi (2-x)(x^3)\,dx$
- $V = \int_{-1}^{1} 2\pi (x+2)(x^3)\,dx$
- $V = \int_{-1}^{1} 2\pi (2-x)(1-x^3)\,dx$
- $V = \int_{-1}^{1} 2\pi (x+2)(1-x^3)\,dx$
- $V = \int_{-1}^{1} \pi (2-x)^2(1-x^3)\,dx$

4. A force of 80 N is required to hold a spring stretched 6 m beyond its natural length. How much work is done in stretching it from its natural length to 10 m beyond its natural length?

- $F = m\cdot g = m\cdot a$
- $W = F\cdot d = \int F(x)\,dx$
- Hooke's Law: $F = kx$
- $\theta_0 = (k)(l_0)$
- $\frac{\theta_0}{l_0} = k$
- $\frac{\theta}{l_0} = K$
- $\frac{\theta}{3} = K$
- $F = 493x$
Evaluate the integral \( \int 3x \cos 2x \, dx \).

By parts:

\[
\int u \, dv = uv - \int v \, du
\]

\[
\begin{align*}
\text{Let } u &= 3x & \quad \text{and } dv &= \cos 2x \, dx \\
\Rightarrow & \quad \frac{du}{dx} = 3 & \quad \Rightarrow & \quad \int v \, du = \int \frac{1}{2} \sin 2x \, dx \\
\end{align*}
\]

\[
\frac{3}{2} \sin (2x) - \frac{3}{2} \sin (2x) + C
\]

What is the volume of the solid obtained by rotating the region between the \( x \)-axis and \( \sin(x) \sqrt{\cos(x)} \) between \( x = 0 \) and \( x = \frac{\pi}{2} \) around the \( x \)-axis?

\[
\begin{align*}
\pi R^2 \\
\pi \int_0^{\frac{\pi}{2}} \left( \sin(x) \sqrt{\cos(x)} \right)^2 \, dx \\
= \pi \int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos(x) \, dx \\
\end{align*}
\]

Change bounds with \( u \)-substitution:

\[
\begin{align*}
\pi \int_0^{\frac{\pi}{2}} u^2 \, du = \pi \left[ \frac{u^3}{3} \right]_0^{\frac{\pi}{2}} \\
= \frac{\pi}{3}
\end{align*}
\]
7. Find the area enclosed by the curves $y = x^2$ and $y = 8x - x^2$.

- $\frac{32}{3}$
- $64$
- $\frac{16}{3}$
- $256$
- $\frac{128}{3}$

\[
\int (8x - x^4) - x^2 \, dx = \int \frac{1}{2} \frac{d}{dx} \left( \frac{x^3}{3} \right)^2 \, dx = \frac{1}{6} \frac{d}{dx} \left( \frac{x^3}{3} \right)^3 + \frac{1}{2} \frac{d}{dx} \left( \frac{x^3}{3} \right)^3
\]

8. Suppose you wish to compute the integral $\int \frac{\sqrt{9 - 4x^2}}{x} \, dx$. Which of the following is the correct trigonometric substitution?

- $x = \frac{a}{b} \sin \theta$
- $x = \frac{b}{a} \sin \theta$
- $x = \frac{a}{b} \sec \theta$
- $x = \frac{b}{a} \tan \theta$
- $x = \frac{a}{b} \sec \theta$
- $x = \frac{b}{a} \tan \theta$
Part II: Justify your answer and show all work for full credit.

Find the area enclosed by the curves $y = \cos x$ and $y = 2 - \cos x$ over the interval $[0, 2\pi]$. 

\[
\int_{0}^{2\pi} \left( (2 - \cos x) - \cos x \right) \, dx
\]

\[
\int_{0}^{\pi} (2 - 2\cos x) \, dx
\]

\[
\left[ 2x + 2\sin(x) \right]_{0}^{2\pi}
\]

\[
\left[ 4\pi + 0 \right] - \left[ 0 + 0 \right]
\]

\[
+ 4\pi
\]
Find the volume of the solid obtained by rotating about \( y \)-axis the region bounded by 
\( y = \sin(x^2) \), \( 0 \leq x \leq \frac{\sqrt{\pi}}{2} \), and \( y = 0 \).

**Cylindrical Shell Method**

\[
\int 2\pi rh 
\]

\[
= \int_{0}^{\frac{\sqrt{\pi}}{2}} 2\pi x \cdot \sin(x^2) \, dx 
\]

\[
u = x^2 \\
\frac{du}{2} = x \, dx \\
\frac{1}{2} \, du = x \, dx
\]

**Change Bands**

\[
2\pi \int_{0}^{\frac{\sqrt{\pi}}{2}} \sin(u) \, du = \pi \left[ -\cos(u) \right]_{0}^{\frac{\sqrt{\pi}}{2}} 
\]

\[
= \pi \left[ -1 - \frac{\sqrt{\pi}}{2} \right]
\]

\[
= \pi \left[ 1 - \frac{\sqrt{\pi}}{2} \right]
\]
A cable that weighs 1/2 lb/ft is used to lift a 10 lb bucket of water from a well 20 ft deep. How much work is done?

\[
\text{Total cable weight: } \int_{0}^{20} \left(20 - \frac{1}{2}x\right) \, dx = 10 \text{ lb.}
\]

\[
\text{Total weight at bottom: bucket + full cable = 10 + 20 = 30 lb.}
\]

in ft, so no gravity constant

\[
\int_{0}^{20} \left(20 - \frac{1}{2}x\right) \, dx = 200 - \frac{1}{4} \left(20^2\right) = 200 - 100 = 100 \text{ ft-lbs}
\]

\[
\text{Subtracting cable weight as it is pulled up:}
\]

\[
\left[20x - \frac{1}{4}x^2\right]_{0}^{20} = 400 - \frac{1}{4}(400) = 400 - 100 = 300 \text{ ft-lbs}
\]
Evaluate the integral \( \int x^2 e^{ax} \, dx \), where \( a \) is a constant.

\[
\int u \, dv = uv - \int v \, du
\]

\[
\begin{align*}
U &= x^2 & dv &= e^{ax} \, dx \\
dU &= 2x \, dx & v &= \frac{1}{a} e^{ax}
\end{align*}
\]

\[
\frac{x^2}{a} e^{ax} - \int \frac{2}{a} x e^{ax} \, dx
\]

Round 2!

\[
\begin{align*}
U &= \frac{2}{a} x & dv &= e^{ax} \, dx \\
dU &= \frac{2}{a} \, dx & v &= \frac{1}{a} e^{ax}
\end{align*}
\]

\[
\frac{x^2}{a} e^{ax} - \left( \frac{2}{a^2} x e^{ax} - \int \frac{2}{a^2} e^{ax} \, dx \right)
\]

\[
= \frac{x^2}{a} e^{ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^3} e^{ax} + C
\]
Evaluate the integral \( \int \tan^3 x \sec^4 x \, dx \).

\[
\int \tan^3 x \cdot \sec^2 x \cdot \sec^2 x \, dx
\]

\[
\int \tan^3 x \cdot (\tan^2 x + 1) \cdot \sec^2 x \, dx
\]

\[
u = \tan x
\]

\[
du = \sec^2 x \, dx
\]

\[
\int u^3 (u^2 + 1) \, du
\]

\[
= \int (u^5 + u^3) \, du
\]

\[
= \frac{u^6}{6} + \frac{u^4}{4} + C
\]

\[
= \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C
\]
Evaluate the integral \[ \int_1^2 \frac{1}{x^2 \sqrt{x^2 + 9}} \, dx \]

**Trig sub**

\[ x = 3 \tan \theta \]
\[ dx = 3 \sec^2 \theta \, d\theta \]

\[ \frac{x}{3} = \tan \theta \]
\[ \tan^{-1} \left( \frac{x}{3} \right) = \theta \]

\[ \sqrt{x^2 + 9} = \frac{\sqrt{15}}{3} \]

\[ \frac{\sqrt{15}}{3} \]

**No bounds temporarily, have to think hard to make \( \theta \) values**

\[ \int \frac{3 \sec^2 \theta \, d\theta}{9 \tan^2 \theta \sqrt{9 \tan^2 \theta + 9}} = \int \frac{3 \sec^2 \theta \, d\theta}{27 \tan^2 \theta \sec \theta} \]

\[ \int \frac{\sec \theta \, d\theta}{9 \tan^2 \theta} = \frac{1}{9} \int \frac{\cos^2 \theta \, d\theta}{\sin^2 \theta} = \frac{1}{9} \int \frac{1}{\sin^2 \theta} \, d\theta \]

\[ du = \sin \theta \]
\[ du = \cos \theta \, d\theta \]

\[ \frac{1}{9} \int \frac{du}{u^2} = -\frac{1}{9} \frac{1}{u} = -\frac{1}{9} \frac{1}{\sin \theta} \]

\[ \frac{1}{9} \cdot \frac{\sqrt{15}}{3} \]

\[ = \frac{1}{9} \left[ \frac{\sqrt{15}}{2} - \frac{\sqrt{10}}{2} \right] \]
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