Math 313 Chapter 4 Review

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Do NOT write on me!

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1 4.1 Euclidean $n$-Space

Important Formulas to Remember

- **Triangle Inequality**: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
- **Cauchy-Schwarz Inequality**: $\vec{u} \cdot \vec{v} \leq \|\vec{u}\| \|\vec{v}\|$
- **Theorem**: $\vec{u} \cdot \vec{v} = \frac{1}{2} \|\vec{u} + \vec{v}\|^2 - \frac{1}{2} \|\vec{u} - \vec{v}\|^2$
- **Pythagorean Theorem**: $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

**Example Problems**

Find two vectors in $\mathbb{R}^2$ with Euclidean norm 1 whose Euclidean inner product with $(3, -1)$ is zero.

Use the Cauchy-Schwarz inequality to prove that for all real values of $a, b,$ and $\theta$,

$$(a \cos \theta + b \sin \theta)^2 \leq a^2 + b^2$$

Determine whether the given vectors are orthogonal.

(a) $\vec{u} = (1, -3, 2), \quad \vec{v} = (4, 2, -1)$
(b) $\vec{u} = (-4, 6, -10, 2), \quad \vec{v} = (2, 1, -2, -9)$

Prove: if $\vec{u}$ and $\vec{v}$ are $nx1$ matrices and $A$ is an $nxn$ matrix, then

$$(\vec{v}^T A^T A \vec{u})^2 \leq (\vec{u}^T A^T A \vec{u})(\vec{v}^T A^T A \vec{v})$$

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$. Show that

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$
2 4.2 Linear Transformations from \( \mathbb{R}^n \) to \( \mathbb{R}^m \)

Common Linear Transformation Matrices

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| Reflection about the y-axis              | \[
\begin{bmatrix}
-1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & -1 \\
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\] |
| Reflection about the x-axis              | \[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
0 & 1 \\
1 & 0 \\
0 & 0 \\
-1 & 0 \\
\end{bmatrix}
\] |
| Reflection about y=x                     | \[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\] |
| Reflection about the xy-plane            | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Reflection about the xz-plane             | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Reflection about the yz-plane             | \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Orthogonal projection on the x-axis       | \[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\] |
| Orthogonal projection on the x-axis       | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 \\
0 & 1 \\
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Orthogonal projection on the xy-plane     | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Orthogonal projection on the xz-plane     | \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |
| Orthogonal projection on the yz-plane     | \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\] |

Rotation through an angle $\theta$

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

CCW rotation of $\theta$ about positive x-axis

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

CCW rotation of $\theta$ about positive y-axis

\[
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

CCW rotation of $\theta$ about positive z-axis

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Contraction/Dilation by a factor $k$ on $\mathbb{R}^2$

\[
\begin{bmatrix}
k & 0 \\
0 & k
\end{bmatrix}
\]

Contraction/Dilation by a factor $k$ on $\mathbb{R}^3$

\[
\begin{bmatrix}
k & 0 & 0 \\
0 & k & 0 \\
0 & 0 & k
\end{bmatrix}
\]

Example Problems

Find the standard matrix for the linear operator $T$ defined by the formula.

(a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

(b) $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$

(c) $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$

Find the standard matrix for the linear operator that rotates by an angle of 270° about the x-axis and then rotates by an angle of 90° about the y-axis and lastly rotates by an angle of 180° about the z-axis.

Find the standard matrix for a rotation of 90° in $\mathbb{R}^2$ followed by a reflection about the line $y = x$.

Show that if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal projection on one of the coordinate axes, then for every vector $\vec{x}$ in $\mathbb{R}^3$, the vectors $T(\vec{x})$ and $\vec{x} - T(\vec{x})$ are orthogonal vectors.
3 4.3 Properties of Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Important Definitions to Remember

Equivalent Statements If $A$ is an $n \times n$ matrix, then the following are equivalent:

- (a) $A$ is invertible
- (b) $A\vec{x} = \vec{0}$ has only the trivial solution.
- (c) The reduced row-echelon form of $A$ is $I_n$
- (d) $A$ is expressible as a product of elementary matrices.
- (e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$
- (f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$
- (g) $\det(A) \neq 0$.
- (h) The range of $T_A$ is $\mathbb{R}^n$.
- (i) $T_A$ is one-to-one.

Properties of Linear Transformations

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$T(c\vec{u}) = cT(\vec{u})$

Example Problems

Determine whether the linear operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator and find $T^{-1}(w_1, w_2)$

(a) $w_1 = -x_2$
   $w_2 = x_1$

(b) $w_1 = 4x_1 - 6x_2$
   $w_2 = -2x_1 + 3x_2$

Determine whether $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear operator.

(a) $T(x, y) = (2x + y, x - y)$

(b) $T(x, y) = (x + 1, y)$

(c) $T(x, y) = (y, y)$

(d) $T(x, y) = (\sqrt{x}, \sqrt{y})$

Find the standard matrix for $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ from the images of the standard basis vectors.

(a) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ reflects a vector about the $xz$-plane and then contracts that vector by a factor of $\frac{1}{5}$
(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ projects a vector orthogonally onto the $xz$-plane and then projects that orthogonally onto the $xy$-plane.

Is a composition of one-to-one linear transformations one-to-one? Justify your conclusion.

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{0}) = \vec{0}$—that is, $T$ maps the zero vector in $\mathbb{R}^n$ into the zero vector in $\mathbb{R}^n$. 
4 4.4 Linear Transformations and Polynomials

Important Definitions to Remember

Example Problems