Math 112
Exam 1
Oct 3-6, 2016

Encode your BYU ID in the grid below.

Instructions

1. Do not write on the barcode area at the top of each page, or near the four circles on each page.

2. Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely.

3. For questions which require a written answer, show all your work in the space provided and justify your answer.

4. Simplify your answers.

5. No books or notes are allowed.

6. There is no time limit on this exam.
Part I: Multiple Choice Questions: (4 points each) Choose the best answer for each multiple choice question. Fill in the box completely for the correct answer.

1. Find the domain of \( f(x) = \frac{\sqrt{3-x}}{(x-2)(x+1)} \).

- \([3, \infty)\)
- \((-1, 2)\)
- \((-\infty, -1) \cup (-1, 2) \cup (2, 3)\)
- \((-1, 2) \cup (2, \infty)\)
- \((-\infty, \infty)\)
- \((-\infty, 3]\)
- \((-\infty, -1) \cup (-1, 2) \cup (2, \infty)\)
- \((-\infty, -1) \cup (-1, 2) \cup (2, 3]\)

2. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side \( x \) at each corner and then folding up the sides as in the figure. Express the volume \( V \) of the box as a function of \( x \) and give the domain of \( V(x) \).

- \( V(x) = x(20-x)(12-x) \) with domain = \((0, 12)\)
- \( V(x) = 12 \cdot 20 \cdot x^2 \) with domain = \((-\infty, \infty)\)
- \( V(x) = x(20-x)(12-x) \) with domain = \((0, 20)\)
- \( V(x) = x(20-2x)(12-2x) \) with domain = \((0, 6)\)
- \( V(x) = x(20-x)(12-x) \) with domain = \((-\infty, \infty)\)
- \( V(x) = x(20-2x)(12-2x) \) with domain = \((0, 12)\)
- \( V(x) = 12 \cdot 20 \cdot x \) with domain = \((-\infty, \infty)\)
- \( V(x) = x(20-2x)(12-2x) \) with domain = \((-\infty, \infty)\)
3. Simplify the expression \( \tan(\sin^{-1} x) \).

- \( \frac{\sqrt{1-x^2}}{x} \)
- \( \frac{1}{\sqrt{1-x^2}} \)
- \( \frac{x}{\sqrt{1-x^2}} \)
- \( \frac{1}{x} \)
- \( 0 \)

\[ \sin^{-1} x = \theta \]
\[ \tan(\theta) = \frac{x}{\sqrt{1-x^2}} \]

4. Which of the following functions has vertical asymptotes at \( x = -3 \) and no other vertical asymptotes and has a horizontal asymptote at \( y = 2 \)?

- \( f(x) = \frac{2x^3 + 2}{x^2 - 2x - 3} \)
- \( f(x) = \frac{2x^2 - 2}{x^2 - 2x - 3} \)
- \( f(x) = \frac{x^2 + 2}{x^2 + 2x - 3} \)
- \( f(x) = \frac{x^2 - 2}{x^2 + 2x - 3} \)
- \( f(x) = \frac{2x + 2}{x^2 - 2x - 3} \)
- \( f(x) = \frac{2x + 2}{x^2 + 2x - 3} \)

Re: 
- \( f(x) \) has vertical asymptotes at \( x = -3 \) and no other vertical asymptotes and has a horizontal asymptote at \( y = 2 \).
5. The figure shows graphs of $f$, $f'$, and $f''$. Identify each curve.

- $f = c$, $f' = b$, $f'' = a$
- $f = b$, $f' = a$, $f'' = c$
- $f = b$, $f' = c$, $f'' = a$
- $f = c$, $f' = a$, $f'' = b$
- $f = a$, $f' = b$, $f'' = c$
- $f = a$, $f' = c$, $f'' = b$

6. A warm can of soda is placed in a cold refrigerator. Let $T(t)$ be the temperature of the soda at time $t$. Which of the following is true about $\frac{dT}{dt}$?

- $\frac{dT}{dt} > 0$ and $\frac{dT}{dt}$ is not constant.
- $\frac{dT}{dt} < 0$ and $\frac{dT}{dt}$ is not constant.
- $\frac{dT}{dt} = 0$ and $\frac{dT}{dt}$ is constant.
- $\frac{dT}{dt} = 0$ and $\frac{dT}{dt}$ is not constant.
- $\frac{dT}{dt} > 0$ and $\frac{dT}{dt}$ is constant.
- $\frac{dT}{dt} < 0$ and $\frac{dT}{dt}$ is constant.
7 A particle's position at given times are listed in the table below.

<table>
<thead>
<tr>
<th>Position (ft)</th>
<th>0</th>
<th>20</th>
<th>50</th>
<th>75</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (min)</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

Over which of the following intervals of time does the particle experience its greatest average velocity?

- [0, 20]
- [0, 10]
- [10, 15]
- [5, 15]  \[
\frac{75-20}{15-5} = \frac{55}{10} = 5.5
\]
- [5, 10]  \[
75-20 = 55 \quad 55 = 5
\]
- [5, 10]  \[
75-20 = 55 \quad \frac{55}{10} = 5.5
\]
- [15, 20]
- [0, 5]
- [10, 20]

8 Find the value of \( \lim_{x \to 3} \frac{4-x}{x-3} \).

- 4
- -3
- 4
- \(-\infty\)
- \(\infty\)
- 1
- -1
- \(-\frac{4}{3}\)
- \(-\infty\)
9. Find the value of $c$ that makes the following function continuous for all real numbers.

$$f(x) = \begin{cases} \quad cx^2 + 3, & \text{if } x < -2 \\ \quad cx - 1, & \text{if } x \geq -2 \end{cases}$$

\[ (-2)^2 + 3 = (-2) - 1 \]
\[ c \cdot 2 + 3 = -2c - 1 \]
\[ 6c = -4 \]
\[ c = \frac{-4}{6} = \frac{-2}{3} \]

10. Evaluate \( \lim_{x \to \infty} \frac{3 - e^x}{5e^x + 2} \)

\[ \frac{3 - \infty}{\infty + 2} = \frac{-\infty}{\infty} \Rightarrow \text{l'Hopital's Rule} \Rightarrow \frac{-e^x}{5e^x} = \frac{-1}{5} \]
11. Find the value of \( \lim_{t \to -2} \frac{t^2 - 4}{2t^2 + 5t + 2} \). 

\[
\frac{(t-2)(t+2)}{(2t+1)(t+2)} = \frac{t-2}{2t+1} = \frac{-4}{-3} = \frac{4}{3}
\]

- \( -\frac{4}{3} \)
- does not exist
- \( \frac{1}{2} \)
- \( \frac{4}{3} \)
- \( \frac{4}{5} \)
- \( -\frac{4}{5} \)
- 0
- \( -\frac{1}{2} \)

12. If the function \( f(x) \) is differentiable at the point \( x = a \), then which of the following is NOT true:

\[
\Rightarrow \text{continuous}
\Rightarrow \text{limit exists}
\]

- \( f'(a) \) equals the instantaneous rate of change of \( y = f(x) \) with respect to \( x \) when \( x = a \).
- \( f(x) \) must have a second derivative at \( x = a \). (not true: \( f(x) = \sqrt{1-x^2} \))
- \( f'(a) \) equals the slope of the tangent line to the curve \( y = f(x) \) at \( x = a \).
- the graph of \( f(x) \) cannot have a corner at \( x = a \).
- \( f(x) \) must be continuous at \( x = a \).
- \( f(x) \) cannot have a vertical asymptote at \( x = a \).
Using the graph of \( f(x) = \frac{1}{x} \) below, determine the largest number \( \delta \) such that 

\[ |x - 1| < \frac{1}{10} \]

whenever \( |x - 1| < \delta \).

14. Which of the following is/are FALSE?

- (i) \( (x + y)^2 = x^2 + y^2 \)
- (ii) \( \sqrt{x + y} = \sqrt{x} + \sqrt{y} \)
- (iii) \( \log(x + y) = \log(x) + \log(y) \)
- (iv) \( \sin(x + y) = \sin(x) + \sin(y) \)

- (i), (ii), and (iii) only
- none
- (i), (ii), and (iv) only
- (i), (iii), and (iv) only
- (i) and (iii) only
- all
- (i), (iii), and (iv) only
- (i) and (ii) only
Part II: Free Response Questions: Neatly write solutions for these problems directly on the exam paper. (Work on scratch paper will not be graded.)

Sketch the graph of \( f(x) = -2e^{-x} + 1 \) on the grid below and answer the items below:

(a) List the y-intercept if it exists. \(-1\)

(b) List any x-intercepts if they exist. \( \ln 2 \)

(c) List any horizontal asymptotes if they exist. \( y = 1 \)

(d) List any vertical asymptotes if they exist. None

\[
0 = 1 - 2e^{-x}
\]

\[
2e^{-x} = 1
\]

\[
e^{-x} = \frac{1}{2}
\]

\[
x = -\ln 2
\]

\[
x = \ln 2
\]
Evaluate \( \lim_{{t \to 0}} \left( \frac{6}{t^2 + 3t} - \frac{2}{t} \right) \).

(You will not receive credit if you use l'Hospital's rule to evaluate the limit.)

\[
\lim_{{t \to 0}} \left( \frac{6}{t(t+3)} - \frac{2}{t} \right) = \lim_{{t \to 0}} \frac{6 - 2(t+3)}{t(t+3)}
\]

\[
\frac{6 - 2t - 6}{t(t+3)} = \frac{-2t}{t(t+3)} = \frac{-2}{t+3} + \frac{-2}{3}
\]
Evaluate \( \lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} \).

(You will not receive credit if you use l'Hospital's rule to evaluate the limit).

\[
\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} = \lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} \\
= \lim_{x \to 7} \frac{(x+2) - 9}{(x-7)(\sqrt{x+2} + 3)} = \lim_{x \to 7} \frac{x-7}{(x-7)(\sqrt{x+2} + 3)} = \frac{1}{\sqrt{x+2} + 3} \\
\lim_{x \to 7} \frac{1}{\sqrt{x+2} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}
\]
Prove that \( \lim_{x \to 0} x^4 \sin \frac{3}{x} = 0 \). State any theorems used. (You will not receive credit if you use l'Hospital's rule to evaluate the limit).

Squeeze Theorem

\[-1 \leq \sin \left( \frac{3}{x} \right) \leq 1\]

Since \( \sin \left( \frac{3}{x} \right) \) changes domain, but not range of \( \sin \) function.

\[-x^4 \leq x^4 \cdot \sin \left( \frac{3}{x} \right) \leq x^4\]

Multiply all parts by \( x^4 \)

\[
\lim_{x \to 0} -x^4 \leq x^4 \cdot \sin \left( \frac{3}{x} \right) \leq \lim_{x \to 0} x^4
\]

Apply limit

\[0 \leq x^4 \cdot \sin \left( \frac{3}{x} \right) \leq 0\]

So, by squeeze theorem,

\[
\lim_{x \to 0} x^4 \sin \left( \frac{3}{x} \right) = 0
\]
Prove that $\ln x = -x$ has at least one real root. State any theorems used and justify why the theorems apply.

We need to find a value that produces an output $0$, and a value that produces an output $20$.

\[ \ln x = -x \]

\[ \ln x + x = 0 \]

When $x = \frac{1}{e}$

\[ \ln \left( \frac{1}{e} \right) + \frac{1}{e} = \ln(e)^{-1} + \frac{1}{e} = -\ln(e) + \frac{1}{e} \]

\[ = -1 + \frac{1}{e} < 0 \]

When $x = e$

\[ \ln(e) + e = 1 + e > 0 \]

Because $\ln(x) + x$ is continuous, by the intermediate value theorem, there must exist at least one real root between $x = \frac{1}{e}$ and $x = e$. 
On the grid below, sketch the graph of a function that satisfies all of the given conditions.

1. \( f(0) = 3 \),  
2. \( \lim_{x \to 0^-} f(x) = 4 \),  
3. \( \lim_{x \to 0^+} f(x) = 2 \),  
4. \( \lim_{x \to 4^-} f(x) = -\infty \),  
5. \( \lim_{x \to 4^+} f(x) = \infty \),  
6. \( \lim_{x \to -\infty} f(x) = -\infty \),  
7. \( \lim_{x \to \infty} f(x) = 3 \).
Use the definition of the derivative to find the derivative of $f(x) = \frac{1}{x^2}$.

(You will not receive credit for using other methods to solve this problem.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

Make common denominators:

$$= \lim_{h \to 0} \frac{(x^2) - (x+h)^2}{h(x^2)(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{h(x^2)(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{(x^2)(x+h)}$$

Apply limit...

$$f'(x) = \frac{-1}{(x^2)^2}$$