Math 113
Exam 3
Nov 18, 2016
(Late Day: Nov 22, 2016)

Encode your BYU ID in the grid below.

Instructions

I) Do not write on the barcode area at the top of each page, or near the four circles on each page.
II) Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely. Multiple choice questions are 5 points each.
III) For questions which require a written answer, show all your work in the space provided and justify your answer.
IV) Simplify your answers.
V) No books, notes, or calculators of any type are allowed.
VI) There is no time limit on this exam.
Part I: Multiple Choice Questions: Mark the correct answer. (5 points each)

1. Determine the radius of convergence of the power series \( \sum_{n=0}^{\infty} \frac{n!2^{n-1}}{4^n} \)

- \( \frac{1}{4} \)
- \( 1 \)
- \( \infty \)
- \( 3 \)
- \( \frac{1}{2} \)
- \( 2 \)
- \( 0 \)

\[ \lim_{n \to \infty} \left| \frac{(n+1)x^{n+1}}{n!} \right| = \left| \frac{x^2}{4} \right| \]

\[ |x^2| \leq 4 \]

\[ |x| \leq 2 \]

2. Determine which ONE of the following series is conditionally convergent (i.e. convergent but not absolutely convergent).

- \( \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{n+1} \) always diverges
- \( \sum_{n=3}^{\infty} \frac{(-1)^n}{n^2 - n} \) always converges \( \frac{1}{n^2} \) by \( p \)-test
- \( \sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right) \) \( \lim_{n \to \infty} \frac{\ln(n)}{n} = 0 \) always diverges by \( \text{lim comp test} \)
- \( \sum_{n=1}^{\infty} \frac{\sin(n)}{3^n} \) always converges - comparison test.
- \( \sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2 + 5n + 1} \) \( \lim_{n \to \infty} \frac{n}{n^2 + 5n + 1} = \frac{1}{n} \) \( \lim_{n \to \infty} \frac{n}{n^2 + 5n + 1} = 1 \) so diverges absolutely.

\( \lim_{n \to \infty} \frac{n}{n^2 + 5n + 1} = 0 \) decreases

So by \( \text{AST, conv.} \)
3. Find all the values of $p$ for which the series
\[\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}\]
is convergent.
- $p \geq 1$
- Diverges for all $p$
- $p \geq 2$
- $p > 1$
- $p > 2$
- $p > e$
- $0 < p \leq 1$
- $0 < p < 1$

4. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$ find the smallest $n$ such that the alternating series estimation theorem ensures a remainder less than .001.
\[\frac{1}{(2n+1)!} < \frac{1}{1000}\]

3
1
5
None, since the series diverges.

2
4
6

5. Let $\sum_{n=0}^{\infty} c_n (x - 3)^n$ be a power series. Suppose we know that the series converges at $x = 6$ and divergent when $x = -2$. Then at which of the following points do we know the series converges?
- $x = 7$
- $x = 9$
- $x = 0$
- $x = 1$
- $x = -1$

Series converge at $x = 3$. $c_n 3^n$ converge.
Which of the following is a false statement?

\[ \begin{array}{c}
\checkmark \quad \text{If } \sum a_n \text{ and } \sum b_n \text{ have positive terms and } \lim_{n \to \infty} \frac{a_n}{b_n} = 1, \text{ then either both series converge or both diverge.} \\
\times \quad \text{Let } \sum a_n \text{ and } \sum b_n \text{ have positive terms. If } \sum a_n \text{ is divergent and } a_n \leq b_n \text{ for all } n, \text{ then } \sum b_n \text{ is divergent.} \\
\times \quad \text{Let } \sum a_n \text{ and } \sum b_n \text{ have positive terms. If } \sum b_n \text{ is divergent and } a_n \leq b_n \text{ for all } n, \text{ then } \sum a_n \text{ is convergent.} \\
\checkmark \quad \text{If } \lim_{n \to \infty} a_n \text{ does not exist or if } \lim_{n \to \infty} a_n \neq 0, \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ is divergent. (Divergence Test)} \\
\checkmark \quad \text{If } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1, \text{ then the sequence } \{a_n\} \text{ is divergent. (Ratio Test)}
\end{array} \]

Exactly one of the following sequences diverges. Determine which one. In each case, the \( n \)-th term of the sequence is given.

\[ \begin{array}{c}
\checkmark \quad a_n = \frac{5n^3 + 2}{n^3 - n} \quad \lim_{n \to \infty} a_n = 5 \\
\times \quad a_n = \frac{4^n}{1 + 7^n} \quad \lim_{n \to \infty} a_n = 0 \\
\times \quad a_n = \cos \left( \frac{n\pi}{n + 3} \right) \quad \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \\
\times \quad a_n = \frac{\ln n}{\ln 2n} \quad \lim_{n \to \infty} a_n = 1 \\
\times \quad a_n = \frac{\tan^{-1} n}{n} \quad \lim_{n \to \infty} a_n = 0 \\
\checkmark \quad a_n = \frac{2n^2 - n}{\sqrt{n^3 + 3n^2}} \quad \lim_{n \to \infty} a_n = \infty \quad \text{DIVERGES} \\
\end{array} \]

\[ \lim_{n \to \infty} \frac{\sqrt[3]{n}}{\sqrt{n + 1}} = \frac{\sqrt[3]{n}}{\sqrt{n}} = \infty \]

\[ \lim_{n \to \infty} \left( \frac{1}{n} \right)^{\frac{n-1}{2}} = \infty \]

\[ \lim_{n \to \infty} \left( \frac{1}{n^3} \right)^{\frac{1}{2}} = \frac{1}{n^3} \quad \text{which is 0 as } n \to \infty \]
Find a power series representation for $f(x) = \ln(1 - 3x)$.

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n-1}(n+1)}
\]

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{(-3)^n(n+1)}
\]

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{3(n+1)}
\]

\[
\sum_{n=0}^{\infty} \frac{-(3n+1)x^{n+1}}{n+1}
\]

\[
\sum_{n=0}^{\infty} \frac{(-3)^nx^{n+1}}{n+1}
\]

\[
\sum_{n=0}^{\infty} \frac{3^nx^{n+1}}{n+1}
\]

\[
\frac{\frac{3}{1-3x}}{1-3x} = -3 \sum (3x)^n
\]

\[
= -3 \sum 3^n x^n
\]

\[
f'(x) = \sum -(3)^{n+1} x^n
\]

\[
\text{antiderivative with respect to } x
\]

\[
f(x) = \sum -(3)^{n+1} x^{n+1}
\]

\[
\frac{1}{n+1}
\]
Part II: Justify your answer and show all work for full credit.

Below are two geometric series. Determine for each one, whether it converges or diverges, and, if convergent, calculate the sum.

1. \[ \sum_{n=1}^{\infty} \frac{4^{2n}}{6^{n-1}} = \sum \left( \frac{16}{6} \right)^n = \sum \left( \frac{8}{3} \right)^n \]
   \[ r = \frac{8}{3} > 1 \]
   \[ \text{Diverges} \]

2. \[ \sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-1}}{3^n} = \sum \frac{5 \cdot 2^{n-1}}{3^n} = \sum \frac{5 \cdot 2^n}{3^n} = \frac{5}{3} \sum \left( \frac{2}{3} \right)^n \]
   \[ r = \frac{2}{3} < 1 \]
   \[ \text{Converges} \]
   \[ \text{Sum} = \frac{\text{first term}}{1 - r} = \frac{\frac{5 \cdot 2}{3}}{1 - \frac{2}{3}} = \frac{\frac{5}{3}}{\frac{1}{3}} = \frac{5}{1} = 5 \]

   \[ \left( \frac{5}{3} \right)^n \]
Estimate an upper bound of the error for the sum of the first 100 terms for the series

\[ \sum_{n=1}^{\infty} \frac{3}{5+n^5}. \]

\[
\text{error} < \int_{100}^{\infty} \frac{3}{5+n^5} \, dn \approx \int_{100}^{\infty} \frac{3}{n^5} \, dn
\]

We always simplify these, because the error will be slightly smaller than the number we find, so it's still a true statement.
Below are series. Determine for each one, whether it converges or diverges. State the test you use and show your work.

1. \[ \sum_{n=1}^{\infty} \frac{2^n}{3^n n^3} \] **Ratio Test**

\[
\lim_{n \to \infty} \left| \frac{\frac{2^{n+1}}{3^{n+1} (n+1)^3}}{\frac{2^n}{3^n n^3}} \right| = \lim_{n \to \infty} \left| \frac{2}{3} \cdot \left(\frac{n}{n+1}\right)^3 \right| = \frac{2}{3} < 1
\]

Converges by Ratio Test

2. \[ \sum_{n=1}^{\infty} \frac{(-2n)^n}{n+3} \] **Not Test to get rid of that power**

\[
\lim_{n \to \infty} \left( \frac{(-2n)^n}{n+3} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{2n}{n+3} \right)^{5}\]

Use l'Hôpital's on inside

\[ 2^5 = 32 \text{ diverges by Root Test} \]
Below are series. Determine for each one, whether it converges or diverges. State the test you use and show your work.

1. \( \sum_{n=1}^{\infty} ne^{-n^2} = \sum \frac{n}{e^{n^2}} \) [Ratio test]
   \[
   \lim_{n \to \infty} \left| \frac{\frac{n+1}{e^{(n+1)^2}}}{\frac{n}{e^{n^2}}} \right| = \lim_{n \to \infty} \left| \frac{\frac{n+1}{e^{n^2}}}{\frac{n}{e^{n^2}}} \right| = \lim_{n \to \infty} \left| \frac{n+1}{n} \right| = 1
   \]
   Thus, it diverges by the ratio test.

2. \( \sum_{n=2}^{\infty} \frac{n+1}{n^3 - 1} \) [Limit comparison test]
   \[
   \lim_{n \to \infty} \frac{n+1}{n^3 - 1} \cdot \frac{1}{n^2} = \lim_{n \to \infty} \frac{n+1}{n^3 - 1} 
   \]
   behave similarly, so because \( \frac{1}{n^2} \) converges (by p-series),
   so does this one.
Find the interval of convergence of the power series \( \sum_{n=2}^{\infty} \frac{(x-2)^n}{\sqrt{n} - 1} \).

\[
\lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{\sqrt{n+1} - 1} \cdot \frac{\sqrt{n} - 1}{(x-2)^n} \right| = \lim_{n \to \infty} \left| \frac{(x-2)}{\frac{\sqrt{n+1}}{\sqrt{n} - 1}} \right| = |x-2| < 1
\]

\(-1 < x-2 < 1\)

\(1 < x < 3\)

Plug in end points to test.

\[\lim_{n \to \infty} \frac{(x-2)^n}{\sqrt{n} - 1} = \frac{(-1)^n}{\sqrt{n} - 1}\]

Converges by AST

\[\lim_{n \to \infty} \frac{(3-x)^n}{\sqrt{n} - 1} = \frac{1}{\sqrt{n} - 1}\]

Diverges by comparison & p-series
Find a power series representation for

\[ f(x) = \frac{1}{(2x - 1)^2} \]

and find the radius of convergence for the series.

\[ f(n) = \frac{1}{(2x - 1)^2} \]
\[ f(n) = \sum_{n=1}^{\infty} \frac{1}{(2x - 1)^2} \]
\[ f(n) = \frac{1}{2} \cdot \frac{1}{1 - 2x} = \frac{1}{2} \sum_{n=0}^{\infty} (2x)^n \]

So, \( f(x) = \sum_{n=1}^{\infty} \frac{1}{(2x - 1)^2} \)

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot x^n \]

Do the ratio test to find radius of convergence.

\[ \lim_{n \to \infty} \left| \frac{2(n+1) x^n}{2^n x^{n-1}} \right| = \left| \frac{2x}{2} \right| = \left| x \right| \leq \frac{1}{2} \]

Take derivative to get back to \( f(x) \)

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x^{n-1} \cdot n \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} (2^{n-1}) x^n \]

\[ f(x) = \frac{3x}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} x^n \]

\[ \text{Radius} = \frac{1}{2} \]