Math 113
Exam 3
Mar 28-30, Late Day Mar 31, 2016

Encode your BYU ID in the grid below.

Instructions

I) Do not write on the barcode area at the top of each page, or near the four circles on each page.

II) Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely. Multiple choice questions are 5 points each.

III) For questions which require a written answer, show all your work in the space provided and justify your answer.

IV) Simplify your answers.

V) No books, notes, or calculators of any type are allowed.

VI) There is no time limit on this exam.
FERPA Permission: Please indicate whether you give permission for your exam to be returned to you by email. This question supersedes any permission you have given previously. Please answer it correctly. No score will be assigned to this question. Note: If you choose not to give permission, you will need to discuss with your instructor how you will get your exam.

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Part I: Multiple Choice Questions: Mark the correct answer. (5 points each)

1. For which (of the following) value of the real number $a$ will the series
\[ \sum_{n=0}^{\infty} \frac{(4a+3)^n}{3a+1.1} \] converge?

☐ -3
☐ 0
☐ 1
☐ 2
☐ -2
☐ -1

\[ \frac{4(-3)^3+3}{3(-3)+1.1} = \frac{-12+3}{-9+1.1} \approx \frac{-7}{-7.9} \approx 1, \text{ DIV} \]

2. Which answer is the sum of the following series?
\[ \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} \]

☐ $\frac{3}{2}$
☐ 3
☐ 2
☐ $\frac{3}{4}$
☐ $\frac{1}{4}$

\[ \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \text{ telescoping sum} \]

\[ \frac{A}{n+1} + \frac{B}{n-1} = \frac{(n-1) + (n+1)}{(n+1)(n-1)} = \frac{2n}{n^2 - 1} \]

\[ \begin{align*}
A &= 1, \\
B &= 1
\end{align*} \]

\[ \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) = 1 \]
3 Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{2^n}{3n} x^n \).

- \( \frac{1}{2} < x \leq \frac{1}{2} \)
- \( -2 < x \leq 2 \)
- \( -2 \leq x < 2 \)
- \( -2 < x < 2 \)
- \( -\frac{1}{2} \leq x \leq \frac{1}{2} \)
- \( -\frac{1}{2} \leq x < \frac{1}{2} \)
- \( \infty < x < \infty \)

4 Determine the behavior of the sequence \( a_n = \cos \left( \frac{n\pi}{n+1} \right) \).

- It does not converge to any real number.
- It converges conditionally to 1.
- It converges to 0.
- It converges to \( \pi \).
- It converges to -1.
- It oscillates between 1 and -1.

5 Which of the following best describes the behavior of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} \)?

- It fails to converge.
- It can't be determined to converge or diverge.
- It is absolutely convergent.
- It is conditionally convergent, but not absolutely convergent.
6. The series \( \sum_{n=0}^{\infty} (-2)^n(x - 3)^n \) converges to which of the following functions on the interval \((2.5, 3.5)\):

- \( \frac{1}{7 - 2x} \)
- \( \frac{2}{x - 1} \)
- \( \frac{2}{1 - x} \)
- \( \frac{1}{1 + (x - 3)/2} \)
- \( \frac{1}{1 - (x - 3)/2} \)
- \( \frac{1}{1 + 2(x - 3)} \)

Geometric series: \( \left[ (-2)(x - 3) \right]^n \)

\[
\begin{align*}
\text{function} & = \frac{\text{sum}}{1 - R} \\
& = \frac{1}{1 + 2x - 6} = \frac{1}{2x - 5}
\end{align*}
\]
Part II: Justify your answer and show all work for full credit.

Determine whether the sequence converges. If it converges, find the limit:

\[ a_n = \left(1 - \frac{2}{n}\right)^n. \]

(Consider using L’Hopital’s rule.)

\[
\lim_{n \to \infty} (1 - \frac{2}{n})^n = \frac{1}{0} \quad \text{indeterminate.}
\]

\[
\lim_{n \to \infty} (1 - \frac{2}{n})^n = L
\]

\[
\lim_{n \to \infty} n \cdot \ln \left(1 - \frac{2}{n}\right) = \ln L
\]

\[
\frac{\ln \left(1 - \frac{2}{n}\right)}{\frac{1}{n}} = \ln L \quad \implies \frac{0}{0} \quad \text{L'Hopital's Rule}
\]

\[
1^{\infty} \rightarrow \lim_{n \to \infty} \frac{\left(\frac{2}{n}\right)^2}{1 - \frac{2}{n}} = \frac{\frac{2}{n^2}}{1 - \frac{2}{n}} \cdot \frac{\frac{n^2}{n}}{1} = \frac{-2}{1 - \frac{2}{n}} = -2
\]

\[
\ln L = -2 \quad \implies L = e^{-2}
\]

Converges to $e^{-2}$.
Determine whether the series is convergent. If it is convergent, find the sum:

\[ \sum_{n=2}^{\infty} \left( \frac{2^n + 4^n}{5^n} \right) \]

\[ = \sum_{n=2}^{\infty} \frac{2^n}{5^n} + \sum_{n=2}^{\infty} \frac{4^n}{5^n} \]

Both converge by geometric series. 

\[
\text{sum} = \frac{\text{1st term}}{1-r}
\]

\[ = \frac{\frac{2}{5^2}}{1 - \frac{2}{5}} + \frac{\frac{4}{5^2}}{1 - \frac{4}{5}} \]

\[ = \frac{\frac{4}{25}}{\frac{3}{5}} + \frac{\frac{16}{25}}{\frac{1}{5}} \]

\[ = \frac{4}{15} + \frac{16}{5} = \frac{4}{15} + \frac{48}{15} = \frac{52}{15} \]
Determine whether the series is absolutely convergent, conditionally convergent, or divergent. State all tests or theorems you use:

\[ \sum_{n=1}^{\infty} (-1)^n \cos \left( \frac{1}{n^2} \right) \]

\[ \lim_{n \to \infty} \left| (-1)^n \cos \left( \frac{1}{n^2} \right) \right| = \lim_{n \to \infty} \cos \left( \frac{1}{n^2} \right) = \cos(0) = 1 \]

Diverges by Divergence Test.
1. Find the error bounds for \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) if the 10th partial sum is used to estimate the sum.

2. For what value of \( n \) is the error bound less than 0.001?

\[
\int_{n}^{\infty} \frac{1}{x^2} dx < R_n < \int_{10}^{\infty} \frac{1}{x^2} dx
\]

\[
\frac{1}{n} < R_n < \frac{1}{10}
\]

\( R_n < \int_{a}^{\infty} \frac{1}{x^2} dx < \frac{1}{1000} \)

\( \frac{1}{n} \alpha^a = \frac{1}{1000} \)

\( \alpha = 100 \)

For 1001 terms, the error will be less than 0.001.
Find a power series for the function \( f(x) = \frac{2x}{1 - x^2} \) and give the interval of convergence.

\[
\begin{align*}
f(x) &= \frac{2x}{1 - x^2} = 2x \cdot \sum (x^2)^n = 2x \sum x^{2n} = \sum 2x^{2n+1} \\
\text{power series:} &\quad \sum 2x^{2n+1} \\
\text{loc:} &\quad \text{Use} \ n \text{-term test} \ \text{for} \ |x| < 1 \\
\frac{2x^{2(n+1)+1}}{2x^{2n+1}} &= |x^2| < 1 \\
|x^2| &\leq 1 \\
-1 &\leq x < 1 \\
\text{loc:} &\quad (-1, 1)
\end{align*}
\]
Find the interval of convergence of \( \sum_{n=2}^{\infty} \frac{(2x-3)^n}{3^n \ln(n) + 1} \).

**Ratio Test**

\[
\lim_{n \to \infty} \left| \frac{(2x-3)^{n+1}}{3^{n+1} \cdot \ln(n+1) + 1} \cdot \frac{3^n \cdot \ln(n) + 1}{(2x-3)^n} \right| = \frac{2x^2}{3}
\]

\[
= \lim_{n \to \infty} \left| \frac{(2x-3)(3^n \cdot \ln(n)) + (2x-3)}{(3^{n+1})(\ln(n+1)) + 3^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(2x-3)3^n \cdot \ln(n)}{3^{n+1}(\ln(n+1)) + 3^{n+1}} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(2x-3)3^n \cdot \ln(n)}{3^{n+1} \cdot \ln(n+1) + 3^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(2x-3) \ln(n)}{3(\ln(n+1) + 1)} \right| = \frac{2x-3}{3} = \frac{2x}{3} \geq 1
\]

**Conclusion:** The interval of convergence is \( [0, \frac{3}{2}] \).
Determine whether the series \( \sum_{n=1}^{\infty} \frac{e^n + 1}{n^2 e^n + n} \) is absolutely convergent, conditionally convergent, or divergent. State all tests or theorems you use.

\[
\sum_{n=1}^{\infty} \frac{e^n + 1}{n^2 e^n + n} = \sum_{n=1}^{\infty} \frac{e^n}{n^2 e^n + n} + \sum_{n=1}^{\infty} \frac{1}{n^2 e^n + n}
\]

if we can show these 2 series converge, then the whole thing converges.

\[
\frac{e^n}{n^2 e^n + n} < \frac{e^n}{n^2 e^n} = \frac{1}{n^2}
\]

So \( \sum \frac{e^n}{n^2 e^n + n} \) converges by comparison test.

\[
\frac{1}{n^2 e^n + n} < \frac{1}{n^2 e^n} < \frac{1}{n^2}
\]

So \( \sum \frac{1}{n^2 e^n + n} \) converges by comparison test.

Therefore the whole series converges.

General rule of thumb—splitting series into smaller, simpler pieces when possible will help you to figure out how to deal with them easier.

- Brinley Fransen