MS Algebra Exam – January 2018

Answer questions 1–8, Part I. Then answer only two of 9–14, Part II. Partial credit will be given.

Part I Answer all the questions 1–8.

1. Let $K = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the Klein 4-group). Prove that $\text{Aut}(K)$ (the group of automorphisms of $K$) is isomorphic to $S_3$.

2. Prove that a group $G$ is Abelian if and only if the map $\theta : g \mapsto g^{-1}$ is a homomorphism.

3. Let $A$ and $B$ be two subgroups of group $G$. Prove if $AB = BA$, where $AB = \{ab | a \in A \text{ and } b \in B\}$, then $AB$ is a subgroup of $G$.

4. In the ring $\mathbb{Z}[z]$, let $I = \langle x, x^2 + 3 \rangle$ (the ideal generated by $x$ and $x^2 + 3$). Prove that $I$ is a maximal ideal.

5. Prove that $\mathbb{Z}_5[x]/(x^2 + 1)$ is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$.

6. Consider the vector space $C_{[-1,1]}$ of continuous functions on $[-1,1]$. For $f$ and $g$ in $C_{[-1,1]}$ let

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx.$$ 

Prove $\langle f, g \rangle$ is an inner product on $C_{[-1,1]}$.

7. Let $A$ be an $n \times n$ matrix there $A^T = A^{-1}$. Prove that the columns of $A$ form an orthonormal basis of of $\mathbb{R}^n$.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation give by $T(x) = Ax$ where $A$ is an $n \times n$ invertible matrix. Prove that $T$ is a one-to-one transformation.
Part II Answer only two of these:

9. Determine the minimum polynomial for
\[
\begin{bmatrix}
5 & 1 & 2 \\
-4 & 0 & -2 \\
-4 & -1 & -1
\end{bmatrix}.
\]

10. Show that if \(AB = BA\) then \(e^{A+B} = e^A e^B\).

11. Let \(F\) be a field. If \(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\) is irreducible in \(F[x]\), prove that \(c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n\) is also irreducible in \(F[x]\).

12. Let \(F\) be a field of order \(p^k\) where \(p\) is a prime. Prove that \(\mathbb{Z}_p\) is a subfield of \(F\), and that \(F\) is a splitting field over \(\mathbb{Z}_p\).

13. Let \(G = C_4 \times C_4\). Prove that there is no irreducible representation \(\sigma\) of \(G\) such that \(g \sigma = (-1)\) for all elements so of \(G\) of order 2 in \(G\).

14. Let \(G\) be the cyclic group of order \(m\), say \(G = \langle a : a^m = 1 \rangle\). Suppose that \(A \in \text{GL}(n, \mathbb{C})\), and define \(\rho : G \to \text{GL}(n, \mathbb{C})\) by
\[
\rho : a^r \to A^r \quad (0 \leq r \leq m - 1).
\]
Show that \(\rho\) is a representation of \(G\) over \(\mathbb{C}\) if and only if \(A^m = I\).