Master's Analysis Exam – January 2018

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. If two real sequences $(a_n) \to a$ and $(b_n) \to b$, show that $(a_n \cdot b_n) \to a \cdot b$.

2. For a real sequence (a_n), show that if $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \geq 0$, then $(a_n) \to 0$.

3. Let A be a nonempty subset of $\mathbb{R}^N$. Show that if every sequence $(a_n) \subset A$ has a convergent subsequence which converges to a point in A (i.e., A is compact), then A is bounded.

4. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions. If (f_n) converges uniformly to f, show that f is continuous.

5. Assume f is differentiable in the open unit ball $B_1 \subset \mathbb{R}^n$ and $|\nabla u| \leq M$ in B_1. If $f(0) = 0$, show that $|f(x)| \leq M|x|$ for any $x \in B_1$.

6. Find the maximum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ for $c \neq 0$.

7. If f has continuous second order partial derivatives in $\mathbb{R}^3$ and also $f_{xx} + f_{yy} + f_{zz} = 0$, use the divergence theorem to show that

$$\int_S \nabla f \cdot \nu \, d\sigma = 0,$$

   where S is the unit sphere (i.e. $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$), and ν is the outward unit normal to S.

8. In n-dimensions, n vectors generate a parallelipiped. Find the volume of the 4-dimensional parallelipiped generated by the vectors

$$\nu_1 = \langle 1,1,1,1 \rangle, \quad \nu_2 = \langle 1,0,1,0 \rangle, \quad \nu_3 = \langle 0,0,1,0 \rangle, \quad \nu_4 = \langle 2,0,0,2 \rangle.$$

9. If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, show that $u_{xx} + u_{yy} = 0$.

10. If C is the circle $|z| = 1/2$ in $\mathbb{C}$, find

$$\oint_C \sum_{k=-10}^{\infty} k z^k \, dz.$$