Algebra Syllabus for Ph.D. Qualifying Examination

1. Group Theory
   (a) Basic definitions. Homomorphisms. Normal subgroups. Lagrange's theorem. Quotient groups.
   (b) Examples: Symmetric groups, dihedral groups, cyclic groups, the quaternion group, etc.
   (c) Group actions.
   (d) Solvable groups. The Jordan-Hölder theorem.
   (e) The Sylow Theorems.
   (f) Direct and semidirect products.
   (g) Free groups and their universal mapping property. Presentations of groups.
2. Ring Theory
   (a) Basic definitions. Homomorphisms. Ideals. Quotient rings.
   (b) Examples. Polynomial rings, matrix rings, etc.
   (c) Rings of fractions
   (d) Chinese remainder theorem
   (e) Euclidean domains, principal ideal domains, and unique factorization domains.
   (f) Unique factorization in polynomial rings. Tests for irreducibility.
3. Module Theory
   (a) Basic definitions. Homomorphisms. Submodules and quotient modules.
   (b) Exact sequences.
   (c) Tensor products.
   (d) Projective, injective, and flat modules.
   (e) Modules over principal ideal domains, with applications to canonical forms for matrices, and the structure theorem for finitely generated abelian groups.
4. Field Theory and Galois Theory
   (a) Field extensions. Algebraic and transcendental extensions.
   (b) Splitting fields. Existence and uniqueness of algebraic closure.
   (c) Separable and inseparable extensions.
   (d) Cyclotomic polynomials and extensions.
   (e) Fundamental theorem of Galois theory
   (f) Finite fields.

(g) Solvable and radical extensions. Application to insolvability of the quintic.
(h) Examples of Galois groups of low-degree polynomials.
(i) Galois-theoretic proof of the fundamental theorem of algebra.
(j) Transcendence bases.

5. Commutative Algebra

   (a) Noetherian rings and modules.
   (b) Hilbert's basis theorem.
   (c) Hilbert's Nullstellensatz.

Most of this material can be found in Dummit-Foote, *Abstract Algebra*, 3rd edition.