MS Algebra Exam — January 2019

Instructions: *Answer all of questions 1–8. Then answer two of questions 9–14. Write your solutions on the front side of the paper only. Leave a one-inch margin on each page.*

Answer all of questions 1–8.

Linear Algebra

1. Let A be a $k \times k$ real matrix whose eigenvalues are all real, distinct, and have absolute value less than 1. Show that each entry of A^n approaches 0 as $n \to \infty$.

2. Let $S = \{s_1, s_2, \ldots, s_m\}$ be a set of vectors in a vector space V, and let $T : V \to V$ be a linear transformation. Show that if the set $T(S) = \{T(s_1), T(s_2), \ldots, T(s_m)\}$ is linearly independent, then the original set S is also linearly independent.

3. If P_2 denotes the space of polynomials of degree less than or equal to 2 with real coefficients, then

$$\big\langle f(x), g(x) \big\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$$

is an inner product on the space. Use the Gram-Schmidt process with the basis

$$B = \{x^2 - 1, x^2 + 1, x^2 + x + 1\}$$

to find a basis for the P_2 which is orthogonal with respect to this inner product.

4. Let a and b be elements of a group G. Prove that the elements ab and ba have the same order.

5. If K is a Sylow p-subgroup of G and H is a subgroup contained in $N(K)$, prove that $[G\colon H] \equiv 1 \pmod{p}$.

   Recall: The normalizer of K is defined by $N(K) = \{g \in G\colon g^{-1}Kg = K\}$.

6. Let G be a finite group. Prove that G is isomorphic to a subgroup of the symmetric group S_n for some $n \in \mathbb{N}$.

7. Prove that the principal ideal $(x^2 - 1)$ is not a maximal ideal in the polynomial ring $\mathbb{R}[x]$

8. Determine the smallest positive integer r such that $2^{10^6} \equiv r \pmod{41}$.

   [Hint: Consider the order of the multiplicative group $(\mathbb{Z}/41\mathbb{Z})^{\times}$.]

Answer two of questions 9–14.

9. Let $\mathbb{F}_3$ denote the finite field of order 3. Find an explicit isomorphism of fields between

$$\mathbb{F}_3[x]/(x^3 - x + 1) \quad \text{and} \quad \mathbb{F}_3[y]/(y^3 + y + 1).$$

10. Determine the splitting field of $x^3 + 6$ over $\mathbb{Q}$, and find all intermediate fields between $\mathbb{Q}$ and the splitting field.

11. Let G be a finite group, and let $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ be a real representation of finite degree. Show that there is some basis B of $\mathbb{R}^n$ such that for each $g \in G$, the matrix $\rho(g)$ with respect to this matrix is block diagonal

$$\rho(g) = \begin{pmatrix} \varphi_1(g) & & & \\ & \varphi_2(g) & & \\ & & \ddots & \\ & & & \varphi_m(g) \end{pmatrix}$$

where each φ_i is an irreducible representation.

12. Determine the character table for the additive group $\mathbb{Z}_4$.

13. Let V be a vector space, T a linear operator that maps V to V, m a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that

$$v, Tv, T^2v, \ldots, T^{m-1}v$$

is linearly independent.

14. Let

$$B = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Compute the eigenprojections P_λ of B.

(a) Write the spectral decomposition $B = \sum_\lambda \lambda P_\lambda + D_\lambda$.

(b) Compute $\cos(B\pi/6)$.

$$\left(\text{Hint: If} A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1/a & -b/(ad) & (be-cd)/(adf) \\ 0 & 1/d & -e/(df) \\ 0 & 0 & 1/f \end{bmatrix}.\right)$$