Master's Analysis Exam – January 2019

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY AND LEAVE AN INCH MARGIN ALL AROUND.

1. Assume that f is a non-decreasing function mapping $\mathbb{R} \to \mathbb{R}$, i.e., that if $x < y$ then $f(x) \leq f(y)$. Show that the number of points x at which f is discontinuous, is either finite or countably infinite.

2. Consider the set $(0, 1]$, an interval on the real line. Produce an open covering $\mathcal{H} = \{H_\alpha : \alpha \in \mathrm{A}\}$ of $(0, 1]$, such that $\mathcal{H}$ has no finite subset that covers $(0, 1]$. Explain why $\mathcal{H}$ has no finite subset that covers $(0, 1]$.

3. Let $f(x) : [-1, 1] \to \mathbb{R}$ be given by $f(x) = \sqrt[3]{x}$. f is uniformly continuous on $[-1, 1]$. According to the definition of uniform continuity, for this particular function f, what is an appropriate choice of δ? Be specific - provide a direct formula that tells us what δ should be.

4. Assume that $f_n : \mathbb{R} \to \mathbb{R}$ satisfies $f'(x) > 0$ if $x > 0$, and $f'(x) < 0$ if $x < 0$, and $f'(0) = 0$. Use the mean value theorem to prove that $f(x) > f(0)$ for $x \neq 0$.

5. Assume that the set $A \subset (0, 1)$ has the following property: That for every $\epsilon > 0$ there is a finite collection of disjoint open intervals $\{\mathcal{I}_n^\epsilon : 1 \leq n \leq N^\epsilon\}$, such that $\mathcal{I}_n^\epsilon = (a_n^\epsilon, b_n^\epsilon) \subset (0, 1)$, and $A \subset \cup_{n=1}^{N^\epsilon} \mathcal{I}_n^\epsilon$, and $\sum_{n=1}^{N^\epsilon}(b_n^\epsilon - a_n^\epsilon) < \epsilon$. (Note: This implies that the set A has measure 0.) Let $f(x) = 1$ if $x \in A$, and let $f(x) = 0$ otherwise. Using the partition-based definition of the Riemann Integral, prove that $\int_0^1 f(x)\,dx$ exists.

6. Let $f : \overline{B}_1 \to \mathbb{R}$ be continuous where $\overline{B}_1$ is the closed unit ball in $\mathbb{R}^n$. Suppose there exists x with $|x| = 1$ and $f(x) > 2$. If $f(0) < 2$, prove there exists a point $y \in B_1$ (the open unit ball) such that $f(y) = 2$.

7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ have continuous second derivatives, and also suppose there exists $\lambda \in \mathbb{R}$ such that $f''(x) = \lambda f(x)$ and $g''(x) = \lambda g(x)$ for all $x \in \mathbb{R}$. If $u(x, y) = f(x)g(y)$, show that
$$\frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \quad \text{for every } (x, y) \in \mathbb{R}^2.$$

8. Evaluate
$$\oint_C F \cdot dr$$

where C is the circle $x^2 + y^2 = 12$ in the plane $z = 0$ with counter clockwise rotation and $F = \langle 2y, -z, x \rangle$.

9. Evaluate

$$\int_C ze^z \, dz$$

where C is the straight line from $1 + i$ to $2 + 3i$.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Let C be a simple closed contour that is smooth. Assuming that f' is continuous, use Green's theorem to prove that

$$\oint_C f(z) \, dz = 0.$$