Ph.D. QUALIFIER EXAMINATION: ANALYSIS

January 2019

Instructions: Answer all the questions given. Each question is graded out of 10. To pass this exam, you need to get 35 out of 60. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY AND LEAVE AN INCH MARGIN ALL AROUND.

Questions

1. State and prove Fatou's Lemma.

2. Consider a sequence of real-valued measurable functions g_n on $[-1,1]$ that converge almost everywhere to a measurable function g on $[-1,1]$. Prove that for any $\epsilon > 0$ there exists a measurable set $E \subset [-1,1]$ with the measure of E smaller than ϵ such that g_n converges uniformly to g on $[-1,1] \setminus E$.

3. Let ℓ be a continuous linear functional on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$. Prove that there exists an element $g \in H$ such that $\ell(f) = \langle f, g \rangle_H$ for all $f \in H$.

4. Let μ be a complex measure on a σ-algebra $\mathcal{M}$. Prove for every $E \in \mathcal{M}$ that
$$|\mu|(E) = \sup \left\{ \left| \int_E f \, d\mu \right| : f \text{ is measurable and } |f| = 1 \right\}.$$

5. Suppose that $f : X \to [0, \infty)$ is a measurable function such that $1/(1+f)$ is Lebesgue integrable. Compute the value of
$$\lim_{n\to\infty} \int_X \frac{\sin(f/n)}{(1+f/n)^n} \, d\mu.$$
Justify your calculation.

6. If $1 \le p < q < r \le \infty$, prove that every $f \in L^p(\mu) \cap L^r(\mu)$ satisfies
$$\|f\|_q \le \|f\|_p^\lambda \, \|f\|_r^{1-\lambda}$$
where $1/q = \lambda/p + (1-\lambda)/r$.