WINTER 2019 - PH.D. PRELIMINARY EXAMINATION ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first 6 that appear. WRITE ON ONE SIDE OF THE PAPER ONLY AND LEAVE AN INCH MARGIN ALL AROUND.

(1) Let $x' = f(x)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitz. Assume that $\xi \in \mathbb{R}^2$ is periodic for the system and there are no periodic points in the interior of orbit of ξ. Prove there is a fixed point in the interior of the orbit of ξ.

(2) Consider the following differential equation in $\mathbb{R}^n$

$$x' = Ax + B(t)x + g(x,t)$$

where
- (a) A is a constant $n \times n$ matrix whose eigenvalues all have negative real parts,
- (b) $B(t)$ maps continuously into the set of $n \times n$ matrices such that $\|B(t)\| \to 0$ as $t \to \infty$, and
- (c) $g(x,t)$ is C^2 and there are constants $a > 0$ and $k > 0$ such that $\|g(x,t)\| \leq k\|x\|^2$ for all $t \geq 0$ and $\|x\| < a$.

Prove that there are constants $C > 1$, $\delta > 0$, and $\lambda > 0$ such that

$$\|x(t, t_0, x_0)\| \leq C\|x_0\|e^{-\lambda(t-t_0)}$$

whenever $t \geq t_0$, $\|x_0\| \leq \delta/C$ and $x(t, t_0, x_0)$ is the solution of the above equation with $x(t_0, t_0, x_0) = x_0$.

(3) Find the fixed points and classify them for the system

$$\begin{aligned} x' &= -10x + 10y \\ y' &= 28x - y - xz \\ z' &= \tfrac{8}{3}z + xy \end{aligned}$$

(4) Consider the flow $e^{At}\mathbf{x}$ for the linear differential equation $\mathbf{x}' = A\mathbf{x}$ with $x \in \mathbb{R}^n$. Prove that all the eigenvalues of A have nonzero real part if and only if for each $\mathbf{x} \in \mathbb{R}^d$ either $\omega(\mathbf{x}) = \{\mathbf{0}\}$ or $\omega(\mathbf{x}) = \emptyset$.

(5) Let $g_\mu(x) = x + x^2 + \mu$ be a one-parameter family of functions with parameter $\mu \in \mathbb{R}$. Show that this family undergoes a saddle-node bifurcation at some $\mu \in (-1, 1)$.

(6) Suppose (X, d_1) and (Y, d_2) are compact metric spaces and $f : X \to X$ and $g : Y \to Y$ are continuous maps. Show that the topological entropy $h_{\text{top}}(f \times g) = h_{\text{top}}(f) + h_{\text{top}}(g)$.

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with a period-2 orbit $f(p_1) = p_2$ and $f(p_2) = p_1$.
(a) Find the Lyapunov exponent $\chi_f(p_1, v)$ of p_1 under the function f for a nonzero $v \in \mathbb{R}$.
(b) Let $g = f^k$ where $k \in \mathbb{N}$. If $\chi_f(x_0, v) = \ell$ for some $x_0 \in \mathbb{R}$, then find $\chi_g(x_0, v)$ in terms of ℓ .

(8) Suppose $f : X \to X$ and $g : Y \to Y$ are continuous functions where X and Y are metric spaces. If there is a semi-conjugacy $\pi : Y \to X$ from g to f, prove that if $g : Y \to Y$ is topologically mixing then $f : X \to X$ is topologically mixing.