WINTER 2019 - PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL
SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first 6 that appear. WRITE ON ONE SIDE OF THE PAPER ONLY AND LEAVE AN INCH MARGIN ALL AROUND.

(1) Let \( x' = f(x) \) where \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is locally Lipschitz. Assume that \( \xi \in \mathbb{R}^2 \) is periodic for the system and there are no periodic points in the interior of orbit of \( \xi \). Prove there is a fixed point in the interior of the orbit of \( \xi \).

(2) Consider the following differential equation in \( \mathbb{R}^n \)

\[
x' = Ax + B(t)x + g(x, t)
\]

where

(a) \( A \) is a constant \( n \times n \) matrix whose eigenvalues all have negative real parts,
(b) \( B(t) \) maps continuously into the set of \( n \times n \) matrices such that \( \|B(t)\| \to 0 \) as \( t \to \infty \), and
(c) \( g(x, t) \) is \( C^2 \) and there are constants \( a > 0 \) and \( k > 0 \) such that \( \|g(x, t)\| \leq k\|x\|^2 \) for all \( t \geq 0 \) and \( \|x\| < a \).

Prove that there are constants \( C > 1, \delta > 0, \) and \( \lambda > 0 \) such that

\[
\|x(t, t_0, x_0)\| \leq C\|x_0\|e^{-\lambda(t-t_0)}
\]

whenever \( t \geq t_0, \|x_0\| \leq \delta/C \) and \( x(t, t_0, x_0) \) is the solution of the above equation with \( x(t_0, t_0, x_0) = x_0 \).

(3) Find the fixed points and classify them for the system

\[
\begin{align*}
x' &= -10x + 10y \\
y' &= 28x - y - xz \\
z' &= \frac{8}{3}z + xy
\end{align*}
\]

(4) Consider the flow \( e^{At}x \) for the linear differential equation \( x' = Ax \) with \( x \in \mathbb{R}^n \).

Prove that all the eigenvalues of \( A \) have nonzero real part if and only if for each \( x \in \mathbb{R}^d \) either \( \omega(x) = \{0\} \) or \( \omega(x) = \emptyset \).

(5) Let \( g_\mu(x) = x + x^2 + \mu \) be a one-parameter family of functions with parameter \( \mu \in \mathbb{R} \).

Show that this family undergoes a saddle-node bifurcation at some \( \mu \in (-1, 1) \).
(6) Suppose \((X, d_1)\) and \((Y, d_2)\) are compact metric spaces and \(f : X \to X\) and \(g : Y \to Y\) are continuous maps. Show that the topological entropy \(h_{\text{top}}(f \times g) = h_{\text{top}}(f) + h_{\text{top}}(g)\).

(7) Suppose \(f : \mathbb{R} \to \mathbb{R}\) is a differentiable function with a period-2 orbit \(f(p_1) = p_2\) and \(f(p_2) = p_1\).
(a) Find the Lyapunov exponent \(\chi_f(p_1, v)\) of \(p_1\) under the function \(f\) for a nonzero \(v \in \mathbb{R}\).
(b) Let \(g = f^k\) where \(k \in \mathbb{N}\). If \(\chi_f(x_0, v) = \ell\) for some \(x_0 \in \mathbb{R}\), then find \(\chi_g(x_0, v)\) in terms of \(\ell\).

(8) Suppose \(f : X \to X\) and \(g : Y \to Y\) are continuous functions where \(X\) and \(Y\) are metric spaces. If there is a semi-conjugacy \(\pi : Y \to X\) from \(g\) to \(f\), prove that if \(g : Y \to Y\) is topologically mixing then \(f : X \to X\) is topologically mixing.