Math 112 (Calculus I)
Final Exam   Form A KEY

Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Approximate $\int_1^5 x^4\,dx$ using a Left Hand sum with 2 subintervals (n=2).

   (a) 82 **(b)** 164 (c) 81 (d) 162 (e) 624 (f) 625
   (g) None of these

2. Find the area under the function $f(x) = \sqrt[3]{x}$ from $x = 1$ to $x = 8$.

   (a) $\frac{45}{4}$ (b) $\frac{1}{4}$ (c) 12 (d) 15 (e) $\frac{1}{12}$ (f) None of these

3. Given the limit statement $\lim_{x\to 1}(2x-3) = -1$ pick the largest δ that works with the definition of the limit if $\epsilon = 0.06$.

   (a) 0.001 (b) 0.005 (c) 0.01 (d) 0.02 **(e)** 0.03 (f) No such δ exists

4. Which of the following is an inflection point of $f(x) = \frac{x}{x^2+1}$?

   (a) 1 (b) -1 (c) 2 (d) -2 (e) $\sqrt{2}$ (f) $-\sqrt{2}$ (g) -3 **(h)** $\sqrt{3}$

5. Given $x\ln y - y\ln x = e^2 - 2e$, find $\frac{dy}{dx}$ at the point (e^2, e).

   (a) 0 (b) e (c) e^2 (d) $\frac{1-e}{e^2}$ **(e)** $\frac{1-e}{e^2-2e}$ (f) e^2-2e (g) $\frac{e-1}{e^2}$

6. Which of the following are x-values for which $f(x) = \sin(x) - x$ has a local maximum?

   (a) -2π (b) $-\pi$ (c) 0 (d) π (e) 2π (f) More than one of these
   (g) None of these

7. Which of the following functions has a discontinuous first derivative?

   (a) $\sinh(x)$ **(b)** $x^{1/3}$ (c) $\tan^{-1}(x)$ (d) $\frac{x}{1+x^2}$ (e) $\ln(x^2+1)$
   (f) All of the first derivatives of these functions are continuous

8. $\frac{d}{dx}\int_1^{2x}\sqrt{1+t^3}\,dt =$

a) $\sqrt{1+(2x)^3}-\sqrt{2}$

b) $2\sqrt{1+(2x)^3}-\sqrt{2}$

c) $\sqrt{1+x^3}-\sqrt{2}]$

d) $2\sqrt{1+x^3}-\sqrt{2}$

e) $\sqrt{1+(2x)^3}$

f) $2\sqrt{1+(2x)^3}$

g) $\sqrt{1+x^3}$

h) $2\sqrt{1+x^3}$

Solution: f)

Short Answer: Fill in the blank with the appropriate answer.

9. (11 points)

(a) Simplify $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \frac{4}{5}$

(b) $\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

(c) $\frac{d}{dx}(\ln(\sin x)) = \frac{\cos(x)}{\sin(x)} = \cot(x)$

(d) $\frac{d}{dx}(\sinh^2(x)) = 2\sinh(x)\cosh(x)$ or $\sinh(2x)$

(e) $\frac{d}{dx}(e^x + x^3) = e^x + 3x^2$

(f) If $f'(x) = e^x + \sin x + x^2$, then $f(x) = e^x - \cos(x) + \frac{x^3}{3} + C$

(g) $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$

(h) $\lim_{x\to 2} \frac{x+2}{x^2-4} =$ Does not exist

(i) $\lim_{x\to 0^+} \sin x - \ln x = \infty$

(j) $\int_1^2 4 + 5x\,dx = \frac{23}{2}$

(k) $\frac{d}{dx}(2^x) = ln(2)2^x$

Free response: Write your answer in the space provided.

10. (8 points)

(a) If $f(x) = \frac{1}{x}$, use the definition of a derivative to set up a limit to find $f'(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

(b) Find $f'(x)$ by evaluating the limit. (No points will be awarded if differentiation rules are used.)

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{-h}{h(x+h)x}$$
$$= \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}.$$

11. (6 points) Find the dimension of the largest rectangle that can be inscribed between the curve $y = 4 - x^2$ and the x-axis.

Solution:

Form A:

The rectangle will have corners at $(-x, 0)$, $(x, 0)$, $(x, 4-x^2)$, and $(-x, 4-x^2)$. Thus, the width of the rectangle is $2x$ and the height is $4 - x^2$. The area therefore is $A(x) = 2x(4 - x^2) = 8x - 2x^3$. Notice that

$$A'(x) = 8 - 6x^2.$$

If we set the derivative to 0, we have $x^2 = \frac{4}{3}$, or $x = \frac{2}{\sqrt{3}}$. Thus, the dimensions are $(\frac{4}{\sqrt{3}}, \frac{8}{3})$.

Form B:

Here the area is $A(x) = 2x(9 - x^2) = 18x - 2x^3$. $A'(x) = 18 - 6x^2 = 0$ gives $x^2 = 3$, so $x = \sqrt{3}$. Thus, the dimensions are $(\sqrt{3}, 6)$.

12. (4 points) $\lim_{x \to 0} \ln(x) \sin(x)$

Solution:

Form A:

$$\lim_{x \to 0} \ln(x) \sin(x) = \lim_{x \to 0} \frac{\ln x}{\csc x}.$$

Use L'Hopital's rule:

$$\lim_{x \to 0} \frac{\ln x}{\csc x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\csc(x)\cot(x)} = \lim_{x \to 0} \frac{\sin x \tan x}{x}$$
$$= \lim_{x \to 0} \cos x \tan x + \sin x \sec^2 x = 0.$$

Form B:

$$\lim_{x \to 0} \ln(x)(1 - \cos(x)) = \lim_{x \to 0} \frac{1 - \cos(x)}{\frac{1}{\ln x}}$$
$$= \lim_{x \to 0} \frac{\sin(x)}{\frac{1}{x \ln^2 x}} = \frac{\cos(x)}{\ln^2 x + 2 \ln x} = 0.$$

13. (3 points) $\dfrac{d}{dx}\left(\ln\left(xe^x - \dfrac{\sin x}{x}\right)\right)$

Solution:

$$\frac{1}{\left(xe^x - \frac{\sin x}{x}\right)}\left(e^x + xe^x - \frac{x\cos x - \sin x}{x^2}\right)$$

14. (4 points) $\displaystyle\int \frac{x}{x^2+4}\,dx$

Solution:

Form A:

Let $u = x^2 + 4$. Then $du = 2x\,dx$ and

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$\int \frac{1}{2}\ln(x^2+4) + C.$$

Form B:

Let $u = x^3 + 9$. Then $du = 3x^2\,dx$ and

$$\int \frac{x^2}{x^3+9}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3+9| + C.$$

15. (10 points) Give the following information about the function $f(x) = x^4 - 4x^3$: (If no information is available in a particular category, leave it blank or cross it out. Putting information in where none exists will be treated as an incorrect answer).

Form A:

All x-intercepts = $(0, 0)$ $(4, 0)$

y-intercept =$(0, 0)$

Intervals for which $f(x)$ is increasing: $(3, \infty)$

Intervals for which $f(x)$ is decreasing: $(-\infty, 3)$

Coordinates of all inflection points: $(0, 0)$ $(2, -16)$

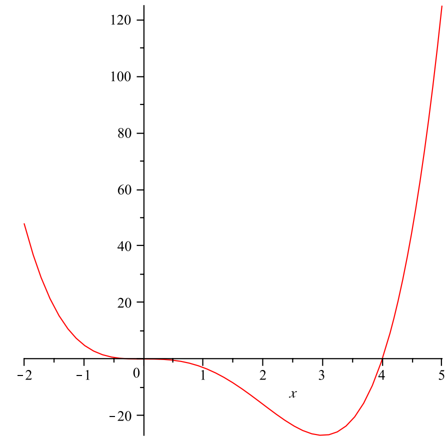
Intervals for which $f(x)$ is concave up: $(-\infty, 0)$, $(2, \infty)$

Intervals for which $f(x)$ is concave down: $(0, 2)$

Coordinates of any local maximums: None

Coordinates of any local minimums: $(3, -27)$

Graph the function:



Form B:

All x-intercepts = $(0, 0)$ $(5, 0)$

y-intercept =$(0, 0)$

Intervals for which $f(x)$ is increasing: $(-\infty, 0)$ $(4, \infty)$

Intervals for which $f(x)$ is decreasing: $(0, 4)$

Coordinates of all inflection points: $(3, -162)$

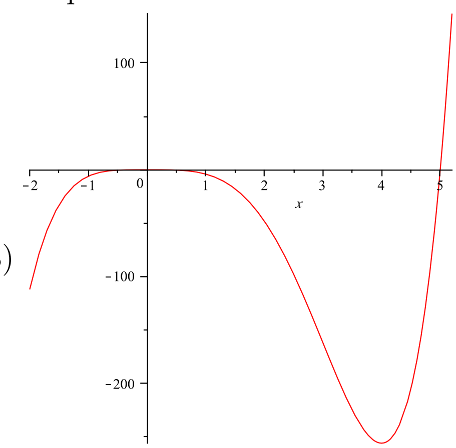
Intervals for which $f(x)$ is concave up: $(3, \infty)$

Intervals for which $f(x)$ is concave down: $(-\infty, 0)$, $(0, 3)$

Coordinates of any local maximums: None

Coordinates of any local minimums: $(4, -256)$

Graph the function:



16. (6 points) A certain element has a half life of 20 years. How many years will it take until only 10% of the element remains? $\left(\text{Note: } \ln\left(\frac{1}{2}\right) \approx -.7 \text{ and } \ln\left(\frac{1}{10}\right) \approx -2.3.\right.$ You can either leave your answer in terms of logs or give a numerical answer using these approximations.$\Big)$

Solution: Since $\frac{1}{2} = e^{20r}$, it is easy to see that $r = \frac{\ln(1/2)}{20} = -\frac{\ln 2}{20}$. For 10 percent to remain, we need

$$.1 = e^{-\frac{\ln 2}{20}t},$$

or

$$\ln(.1) = -\frac{\ln 2}{20}t.$$

Hence,

$$t = -\frac{\ln 2}{20\ln(.1)} = \frac{\ln 2}{20\ln(10)}.$$

17. (6 points) The equation of the tangent line to the curve $y = \frac{1}{x^2}$ at $\left(2, \frac{1}{4}\right)$.

Solution:

Form A:

Note that $y' = \frac{-2}{x^3}$. So, $y'(2) = -\frac{1}{4}$. Thus,

$$y - \frac{1}{4} = -\frac{1}{4}(x-2),$$

and

$$y = -\frac{1}{4}x + \frac{3}{4}.$$

Form B:

$y' = \frac{-3}{x^4}$. $y'(\frac{1}{2}) = -48$.

$$y - 8 = -48(x - \frac{1}{2})$$

or

$$y = -48x + 32.$$

18. (6 points) Use linear approximation to estimate $\sqrt{63}$:

Solution: Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$.

$$\sqrt{63} = \sqrt{64} + \frac{1}{2\sqrt{64}}(-1)$$

$$8 - \frac{1}{16} = \frac{127}{16}.$$

19. (6 points) A pump is blowing up a spherical balloon with a pump rate of $10\text{cm}^3/\text{sec}$. How fast is the diameter of the balloon growing when the balloon has a 5cm radius? $\left(\text{Volume of a sphere is given by } \frac{4}{3}\pi r^3.\right)$

Solution:

Form A:

Differentiate $V = \frac{4}{3}\pi r^3$ to get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Thus,

$$10 = 4\pi \cdot 25 \frac{dr}{dt},$$

and

$$\frac{dr}{dt} = \frac{1}{10\pi}.$$

Form B:

$$10 = 4\pi \cdot 9 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{18\pi}.$$

20. (6 points) A particle is moving with the given data. Find the position function of the particle.

$$a(t) = \sin(t) + 3\cos(t), \quad s(0) = 0, \quad v(0) = 2.$$

Solution:

$$v(t) = -\cos(t) + 3\sin(t) + C$$
$$v(0) = -1 + C = 2$$

so $C = 3$. Thus,

$$v(t) = -\cos(t) + 3\sin(t) + 3.$$
$$s(t) = -\sin(t) - 3\cos(t) + 3t + D$$
$$s(0) = -3 + D = 0, \quad D = 3$$
$$s(t) = -\sin(t) - 3\cos(t) + 3t + 3.$$

END OF EXAM