Math 112 (Calculus I)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. $\dfrac{d}{dx}\displaystyle\int_3^{4x} \frac{t^3}{\sqrt{1+t^5}}\,dt =$

   a) $\dfrac{256x^3}{\sqrt{1+1024x^5}}$ b) $\dfrac{256x^3}{\sqrt{1+1024x^5}} - \dfrac{27}{\sqrt{244}}$ c) $\dfrac{64x^3}{\sqrt{1+1024x^5}}$

   d) $\dfrac{4x^3}{\sqrt{1+x^5}}$ e) $\dfrac{4x^3}{\sqrt{1+x^5}} - \dfrac{27}{\sqrt{244}}$ f) $\dfrac{x^3}{\sqrt{1+x^5}}$

   Solution: a)

2. $\displaystyle\int_{\sqrt{5}}^{2\sqrt{3}} \frac{z}{(4+z^2)^{3/2}}\,dz =$

   a) $-\dfrac{1}{7}$ b) $-\dfrac{1}{12}$ c) Does not exist.

   d) $\dfrac{1}{12}$ e) $\dfrac{1}{7}$ f) $\dfrac{1}{4}$

   g) None of the above.

   Solution: d)

3. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs. (All answers are in square cm.)

   a) 1 b) $\dfrac{1}{3}$ c) 2

   d) 3 e) $\dfrac{9}{2}$ f) There is no largest rectangle.

   g) None of the above.

   Solution: d)

4. $\lim_{x \to 4} \frac{4-x}{|4-x|} =$

a) 0

b) 1

c) -1

d) ∞

e) $-\infty$

f) Does not exist.

g) None of the above.

Solution: f)

5. We say $\lim_{x \to a} f(x) = L$ if

a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

b) There exists $\epsilon > 0$ such that for every $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon$.

c) For some $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

d) For every $\epsilon > 0$ and for every $\delta > 0$, if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

e) For every $\delta > 0$ there exists a $\epsilon > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

f) For some $\epsilon > 0$ and every $\delta > 0$, if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

g) None of the above.

Solution: a)

6. If $f'(x) = e^{x^2}$ and $g(x) = f(\sqrt{x})$, then $g'(x) =$

a) $\frac{e^{x^2}}{2x}$

b) $\frac{e^{x^2}}{2\sqrt{x}}$

c) $\frac{e^{x}}{2\sqrt{x}}$

d) $\left(\frac{2x-1}{4x\sqrt{x}}\right) e^x$

e) $2xe^{\sqrt{x}}$

f) $2\sqrt{x}e^x$

g) None of the above.

Solution: c)

7. $\frac{d}{dx}\left(x^{\sin x}\right) =$

a) $\sin(x)x^{\sin x - 1}$

b) $\cos(x)x^{\sin x - 1}$

c) $x^{\sin x} \cos x \ln x$

d) $\frac{\sin x}{x} x^{\sin x} \cos x$

e) $x^{\sin x}\left(\frac{\sin x}{x} + \cos(x) \ln x\right)$

f) $x^{\sin x} \sin x \cos x$

g) None of the above.

Solution: e)

8. If \$5000 is borrowed at 5% interest compounded continuously, then the amount due at the end of ten years is

a) $\$5000(1.05)^{10}$ b) $\$5000\sqrt{e}$ c) \$7500

d) $\$5000e$ e) $\$5000e^{1.5}$ f) $\$5000e^{1.05}$

g) None of the above.

Solution: b)

Short Answer Fill in the blank with the appropriate answer.

9. (11 points)

a) $\lim_{x \to \pi^-} \ln(\sin x) = -\infty$

b) What kind of discontinuity exists at $x = -1$ for the function $f(x) = \frac{x+1}{x^2-1}$?**removable.**

c) $\frac{d}{dx}(a^3 + \cos^3 x) = \mathbf{3\cos^2 x \sin x}$

d) $\frac{d^2}{dx^2}(e^{x^2}) = \mathbf{(2 + 4x^2)e^{x^2}}$

e) If $f'(x) \geq 2$ for all $x \in [0, 2]$, what theorem tells us that $f(2) - f(0) \geq 4$?**Mean Value Theorem**

f) $\frac{d}{dx}(\tan^{-1}(x^2)) = \mathbf{\frac{2x}{1 + x^4}}$

g) $\lim_{x \to 0^+} \frac{\ln(1+x)}{x} = \mathbf{1}$

h) $\int (\sqrt{x} + \frac{1}{x})\, dx = \mathbf{\frac{2}{3}x^{3/2} + \ln x + C}$

i) $\int_3^4 (1 + 3x)\, dx = \mathbf{\frac{23}{2}}$

j) $\int_2^5 (2x - 1)^2\, dx = \mathbf{117}$

k) Set up a limit to find the derivative of $g(x) = \frac{1}{x^2+1}$. $\lim_{h \to 0} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h}$

Free Response. For problems 10 - 17, write your answers in the space provided. Use the back of the page if needed, indicating that fact. Neatly show all work.

10. (7 points) Prove $\lim_{x \to 2} 4x - 3 = 5$ using the $\epsilon - \delta$ definition of the limit.

Solution: Form A:
Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/4$. Then, if

$$0 < |x - 2| < \delta,$$

we have

$$|x - 2| < \epsilon/4$$

or

$$|4x - 8| < \epsilon.$$

Since $4x - 8 = 4x - 3 - 5$, $|4x - 3 - 5| < \epsilon$, and we are done.

Form B:
Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/2$. Then, if

$$0 < |x + 1| < \delta,$$

we have

$$|x + 1| < \epsilon/2$$

or

$$|2x + 2| < \epsilon.$$

Since $2x + 2 = 2x + 5 - 3$, $|2x + 5 - 3| < \epsilon$, and we are done.

11. (7 points) Estimate $\sqrt[3]{8.012}$ by linear approximation.

Solution: $f(x) = \sqrt[3]{x}$, so $f'(x) = \frac{1}{3x^{2/3}}$. Thus,

$$\sqrt[3]{8.012} = f(8) + f'(8) * 0.012 = 2 + \frac{1}{12}0.012 = 2.001.$$

12. (7 points) Find an equation to the tangent line to $4x^3 + 2xy + y^3 = 1$ at $(1, -1)$.

Solution:
Form A:
Differentiating, we have

$$12x^2 + 2y + 2xy' + 3y^2y' = 0.$$

Since $x = 1$ and $y = -1$, we have

$$12 - 2 + 2y' + 3y' = 0,$$

or $y' = -2$. Plugging this into the point-slope form of the line, we have $y + 1 = -2(x - 1)$ or $y = -2x + 1$.

Form B:
Differentiating, we have

$$6x^2 + y + xy' + 3y^2y' = 0.$$

Since $x = 1$ and $y = -1$, we have

$$6 - 1 + y' + 3y' = 0,$$

or $y' = -\frac{5}{4}$. Plugging this into the point-slope form of the line, we have $y + 1 = -\frac{5}{4}(x - 1)$ or $y = -\frac{5}{4}x + \frac{1}{4}$.

13. (7 points) If a ball is thrown vertically upwards from the top edge of a 90 foot building with an initial velocity of 80 feet per second, it's height above the ground (after t seconds) is given by

$$h(t) = 90 + 80t - 16t^2.$$

What is its maximum height above the ground?

Solution:

$$h'(t) = 80 - 32t = 0$$

means $t = \frac{5}{2}$. $h(5/2) = 190$ feet.

14. Let $f(x) = 2x^3 + 3x^2 - 12x$.

(a) (3 points) Find the interval(s) on which $f(x)$ is increasing or decreasing. Label them appropriately.

Solution: Increasing on $(-\infty, -2)$, $(1, \infty)$
Decreasing on $(-2, 1)$.

(b) (3 points) Find the local maxima and minima of $f(x)$

Solution: Maximum of 20 at -2
Minimum of -7 at 1

(c) (3 points) Find the intervals of concavity and the inflection points of $f(x)$.

Solution: Concave down on $(-\infty, -\frac{1}{2})$
Concave up on $(-\frac{1}{2}, \infty)$ Inflection point at $x = -\frac{1}{2}$.

15. Evaluate the following limits:

(a) (7 points) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

Solution:

Form A:

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x}.$$

Divide top and bottom by x to get

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}.$$

Form B:

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x}-x)(\sqrt{x^2+4x}+x)}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x}.$$

Divide top and bottom by x to get

$$\lim_{x\to\infty} \frac{4}{\sqrt{1+4\frac{1}{x}}+1} = \frac{4}{2} = 2.$$

(b) (7 points) $\lim_{x\to\infty} x^{1/x}$.

Solution: Notice that

$$x^{1/x} = e^{\frac{\ln x}{x}}.$$

Also,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus,

$$\lim_{x\to\infty} x^{1/x} = e^0 = 1.$$

16. (7 points) Find $\int \frac{1+x}{x+x^2}\,dx$.

Solution:

$$\int \frac{1+x}{x+x^2}\,dx = \int \frac{1+x}{x(1+x)}\,dx \int \frac{1}{x}\,dx$$
$$= \ln|x| + C.$$

17. (7 points) A dog owner has 1000 feet of fencing and wishes to make 4 dog runs side by side. (a dog run is a fenced rectangular area the dog can pace in). What dimensions will give the largest area? (Note: two dog runs that sit side by side share a common side.)

Solution:
Form A:
Let x be the width of the 4 dog runs and y be their length. Then, $2x+5y$ is the total amount of fencing. Since this must be 1000 feet, we have

$$2x + 5y = 1000,$$

or

$$y = 200 - \frac{2}{5}x.$$

We wish to maximize $A = xy = x(200 - \frac{2}{5}x) = 200x - \frac{2}{5}x^2$.

$$A' = 200 - \frac{4}{5}x = 0.$$

Thus, $x = 250$, and $y = 100$. Notice that $A''(x) = -\frac{4}{5}$, so the second derivative tells us we have a maximum at this point.

Form B:
Let x be the width of the 3 dog runs and y be their length. Then, $2x+4y$ is the total amount of fencing. Since this must be 800 feet, we have

$$2x + 4y = 800,$$

or

$$y = 200 - \frac{1}{2}x.$$

We wish to maximize $A = xy = x(200 - \frac{1}{2}x) = 200x - \frac{1}{2}x^2$.

$$A' = 200 - x = 0.$$

Thus, $x = 200$, and $y = 100$. Notice that $A''(x) = -1$, so the second derivative tells us we have a maximum at this point.

END OF EXAM