Math 112 (Calculus I)
Final Exam KEY

Short Answer Fill in the blank with the appropriate answer.

1. (10 points)

a) $\dfrac{d}{dx}\ln(\tan x) = \underline{\dfrac{\sec^2 x}{\tan x}}$

b) Use the linearization of $f(x) = x^{1/3}$ at $a = 8$ to approximate $9^{1/3}$. $\underline{\dfrac{25}{12}}$

c) If $f'(x) = x^3$ and $f(0) = 5$ then $f(x) = \underline{\dfrac{x^4}{4} + 5}$.

d) If $f(x) = e^{2x}$ then $f''(x) = \underline{4e^{2x}}$.

e) The Mean Value Theorem says that if f is a <u>continuous</u> function on $[a, b]$

which is also <u>differentiable</u> on (a, b) then there is a $c \in (a, b)$

with $\underline{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$.

f) Circle the correct answer in both cases: If f' is positive and increasing, then f is (**[increasing]** / **decreasing**) and (**[concave up]** / **concave down**).

g) $\lim_{x\to\infty} \frac{x^3+5}{2x^3+3x+2} = \underline{\dfrac{1}{2}}$.

Multiple Choice. In the grid below fill in the square corresponding to each correct answer.

2	A	B	C	**D**	E	F	G	H	I	J
3	A	B	C	**D**	E	F	G	H	I	J
4	A	B	C	D	E	**F**	G	H	I	J
5	A	B	C	**D**	E	F	G	H	I	J
6	A	B	C	D	**E**	F	G	H	I	J
7	A	B	C	D	**E**	F	G	H	I	J
8	A	**B**	C	D	E	F	G	H	I	J
9	A	B	**C**	D	E	F	G	H	I	J
10	A	**B**	C	D	E	F	G	H	I	J

2. If $\sin(xy) = \dfrac{x-1}{y}$, what is y' at $(1, \pi)$?

a) π
b) $\pi^2 - 1$
c) $\pi - \dfrac{1}{\pi}$
d) $\dfrac{-\pi^2 - 1}{\pi}$
e) $-1 - \dfrac{1}{\pi}$
f) $-\dfrac{1}{\pi}$

3. Which of the following has a removable discontinuity at $x = 1$?

a) $\dfrac{x^2 - 1}{x + 2}$
b) $\ln|x^2 - 1|$
c) $\dfrac{x^2 + 2}{x - 1}$
d) $\dfrac{x^2 - 3x + 2}{x^2 - 1}$
e) $\tan \dfrac{\pi x}{2}$
f) $\sin \dfrac{1}{1 - x}$

4. If the position of a particle is given by $\dfrac{t^2 - t}{t - 1}$, at what value(s) of t is the velocity 0?

a) 1
b) $1 \pm \sqrt{2}$
c) $2 \pm \sqrt{2}$
d) $1 - 2\sqrt{2}$
e) 0
f) The velocity is never 0.

5. If $f(x)$ is a differentiable function, which of the following is not always true?

a) $\int_a^x f'(t)dt = f(x) - f(a)$

b) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

c) $\int f'(x)dx = f(x) + C$

d) $\int_a^x F(t)dt = F'(x) - F'(a)$

e) If $F'(x) = f(x)$ then $\int_a^x f(t)dt = F(x) - F(a)$

6. A 20 ft ladder is placed against a wall. If the top of the ladder is dropping at a rate of 5 ft/min, how fast is the bottom of the ladder moving away from the wall (in feet per minute) when the top of the ladder is 12 ft high?

a) -32

b) -160

c) 160

d) $\frac{15}{2}$

e) $\frac{15}{4}$

f) $\frac{15}{16}$

7. If $h(x) = x\cosh(x^2 - 4)$, what is $h''(2)$?

a) 0

b) 4

c) 8

d) 16

e) 32

f) 48

8. Let $F(x) = \int_3^{2x^2} \frac{\sin(t)}{t^3+1}\,dt$ be defined for $x > -1$. Find $F'(x)$.

a) $\frac{\sin(2x^2)}{8x^3+1}$

b) $\frac{4x\sin(2x^2)}{8x^6+1}$

c) $\frac{\sin(2x^2)}{8x^3+1} - \frac{\sin(18)}{216}$

d) $\frac{4x\sin(2x^2)}{8x^3+1} - \frac{12\sin(3)}{28}$

e) $\frac{\sin(x)}{x^3+1}$

f) $\frac{\sin(x)}{x^3+1} - \frac{\sin(3)}{28}$

Solution: b)

9. If $f(x)$ is a continuous function on the interval $[4, 5]$ and if $f(4) = 3$ and $f(5) = 1$, then which theorem guarantees that there is some value $c \in [4, 5]$ such that $f(c) = 2$?

a) Mean value Theorem,

b) Extreme Value Theorem,

c) Intermediate Value Theorem

d) The Fundamental Theorem of Calculus

e) Rolle's Theorem

f) No theorem guarantees this because it is false.

Solution: c)

10. Use one iteration of Newton's method, beginning with $x_1 = 1/2$ to approximate the positive root of the equation $x^2 + 2x - 1 = 0$. (Note that the root is $\sqrt{2} - 1$).

a) $\frac{1}{12}$ b) $\frac{5}{12}$ c) 0

d) $\frac{1}{2}$ e) $\frac{7}{12}$ f) x_2 is undefined.

Free Response. For problems 11 - 19, write your answers in the space provided. Use the back of the page if needed, indicating that fact. Neatly show all work.

11. (7 points) State the definition of the derivative of $f(t)$, and use the *definition* to find the derivative of $f(t) = 2t^2 + 1$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}.$$

The other definition is ok as well.

$$f'(t) = \lim_{h \to 0} \frac{2(t+h)^2 + 1 - (2t^2 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{2t^2 + 4th + 2h^2 - 2t^2}{h} = \lim_{h \to 0}(4t + h) = 4t.$$

12. (7 points) Find the equation of the tangent line to $y = x\sqrt{x+1}$ at the point $(3, 6)$.

Solution:

$$y' = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$$

$$y'(3) = 2 + \frac{3}{4} = \frac{11}{4}$$

$$y - 6 = \frac{11}{4}(x - 3)$$

$$y = \frac{11}{4}x - \frac{9}{4}$$

13. (7 points) Find the integral $\int_1^3 \frac{x}{2x^2 + 1} dx$.

Solution: Let $u = 2x^2 + 1$. Then $du = 4x\, dx$. The integral becomes

$$\int_3^{19} \frac{1}{4u}\, du = \frac{1}{4} \ln u|_3^1 9 = \frac{\ln(19) - \ln(3)}{4}.$$

14. (7 points) $\frac{d}{dx} x^{\cos x} =$

Solution: If $y = x^{\cos x}$, then $\ln y = \cos x \ln x$. Differentiating, we have

$$\frac{1}{y} y' = -\sin x \ln x + \frac{\cos x}{x}.$$

Thus,

$$y' = (\frac{\cos x}{x} - \sin x \ln x) x^{\cos x}.$$

15. (7 points) Find the absolute minimum and maximum of $f(x) = x^2 e^{-x^2/2}$ on $[0, 4]$.

Solution:

$$f'(x) = 2xe^{-x^2/2} - x^3 e^{-x^2/2} = x(2 - x^2)e^{-x^2/2}.$$

Notice that there are two critical points of interest in this interval, 0 and $\sqrt{2}$. Also, it is clear that on the interval $(0, \sqrt{2})$ f' is positive, so f is increasing. On the interval, $(\sqrt{2}, 4)$ f' is negative, so f is decreasing. Hence, the maximum value of the function occurs at $\sqrt{2}$ and is

$$\frac{2}{e}.$$

The minimum must occur at the end points. The smallest value occurs at 0, so the minimum is 0.

16. (7 points) Find $\lim_{x \to 0} \frac{(1-x)e^x - 1}{x^2}$.

Solution:

$$\lim_{x \to 0} \frac{(1-x)e^x - 1}{x^2} = \lim_{x \to 0} \frac{-e^x + (1-x)e^x}{2x} = \lim_{x \to 0} \frac{-e^x - e^x + (1-x)e^x}{2} = \frac{-1}{2}.$$

17. (7 points) An open-at-the-top vertical tank has horizontal cross-section of an equilateral triangle, and volume of 4000 cubic feet. Find the dimensions that minimize the surface area. (Hint: the height of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times the length of a side.)

Solution: The area of the equilateral base is

$$\frac{1}{2}\frac{\sqrt{3}}{2}s \cdot s = \frac{\sqrt{3}}{4}s^2.$$

Thus, the volume of the tank is

$$\frac{\sqrt{3}}{4}s^2 h = 4000.$$

Solving for h, we have

$$h = \frac{16000}{\sqrt{3}s^2}.$$

The surface area of the tank is given by

$$S = \frac{\sqrt{3}}{4}s^2 + 3sh = \frac{\sqrt{3}}{4}s^2 + 3s \cdot \frac{16000}{\sqrt{3}s^2}$$

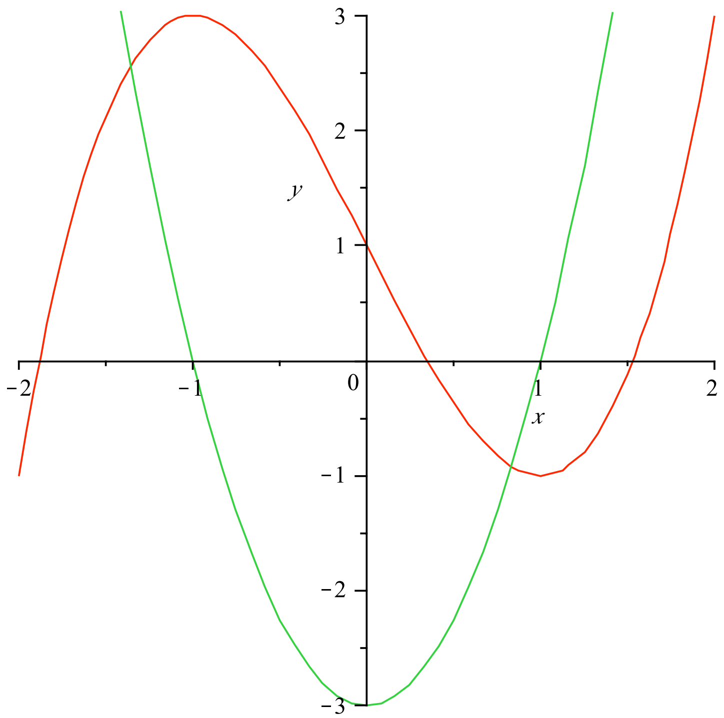
$$= \frac{\sqrt{3}}{4}s^2 + \frac{16000\sqrt{3}}{s}.$$

Thus,

$$S' = \frac{\sqrt{3}}{2}s - \frac{16000\sqrt{3}}{s^2} = \sqrt{3}\frac{s^3 - 32000}{2s^2}.$$

Setting $S' = 0$ we have $s^3 = 32000$ or $s = 20\sqrt[3]{4}$. Hence, $h = \frac{16000}{\sqrt{3}400 \cdot 4^{2/3}} = \frac{40}{\sqrt{3}4^{2/3}}$.

18. (7 points)Given the following graph for $f(x)$, sketch f'.



19. (7 points) Prove that $\lim_{x \to a} 2x = 2a$, using the ϵ, δ definition of limit.

Solution: Let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{2}$. Thus, if

$$0 < |x - a| < \delta,$$

$$|x - a| < \frac{\epsilon}{2}$$

$$\Rightarrow 2|x - a| < \epsilon$$

or

$$|2x - 2a| < \epsilon.$$

By the definition of limit, we are done.

END OF EXAM