The Inverse Eigenvalue Problem for Graphs of Low Minimum Rank

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Brigham Young University
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

\[
A = \begin{bmatrix}
    b & a & c & d \\
    a & d & 0 & 0 \\
    c & 0 & b & 0 \\
    d & 0 & 0 & d
\end{bmatrix} \in S_n(G)
\]
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Given $A \in S_n$, let $G(A)$ be the graph with

vertex set $V = \{1, 2, \ldots, n\}$
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A = \begin{bmatrix}
d_1 & a & b & 0  
da & d_2 & c & 0  
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0 & 0 & d & d_4
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Let $mr(G)$ be the minimum rank over all matrices in $S(G)$.
Minimum Rank Problem

- Let $mr(G)$ be the minimum rank over all matrices in $S(G)$.
- Let $M(G)$ to be the maximum nullity over all matrices in $S(G)$. 

Since $mr(G) + M(G) = n$, computing the minimum rank and the maximum nullity are equivalent problems.

Computing $M(G)$ or $mr(G)$ for a general graph is hard. Easy for $n < 6$.

$mr(paw) = 2$

$\begin{bmatrix}
  d & a & b & 0 \\
  a & d & 2 & c \\
  b & c & d & 0 \\
  0 & 0 & d & d
\end{bmatrix}$
Minimum Rank Problem

- Let \( \text{mr}(G) \) be the minimum rank over all matrices in \( S(G) \).
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$$mr(\text{paw}) = 2$$

$$\begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & 0 \\ b & c & d_3 & d \\ 0 & 0 & d & d_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
The problem of computing maximum nullity is related to a harder problem.
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Question

Given a graph $G$ on $n$ vertices and numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, is there an $A \in S(G)$ such that the eigenvalues of $A$ are exactly these numbers?
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The Combinatorial Inverse Eigenvalue Problem is currently too difficult to solve for almost all graphs.
The problem of computing maximum nullity is related to a harder problem.

**Question**

Given a graph $G$ on $n$ vertices and numbers $\lambda_1, \lambda_2, ..., \lambda_n$, is there an $A \in S(G)$ such that the eigenvalues of $A$ are exactly these numbers?

The Combinatorial Inverse Eigenvalue Problem is currently too difficult to solve for almost all graphs.

But knowing the maximum nullity allows us to at least know the largest possible multiplicity of any eigenvalue of a matrix in $S(G)$. 
Example

Can nonzero $a$, $b$, $c$, $d$ be chosen so that the eigenvalues of
\[
\begin{pmatrix}
  1 & 2 & 3 & 4 \\
  2 & 1 & 4 & 3 \\
  3 & 4 & 1 & 2 \\
  4 & 3 & 2 & 1
\end{pmatrix}
\]
are 3, 2, 1, 0?

Answers: 3 Yes 1 No
Example

Can nonzero $a, b, c, d$ be chosen so that the eigenvalues of

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are $3, 2, 1, 0$?  $2, 1, 0, 0$?
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are 3, 2, 1, 0?  2, 1, 0, 0?  1, 1, 0, 0?  1, 0, 0, −1?
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are $3, 2, 1, 0$? $2, 1, 0, 0$? $1, 1, 0, 0$? $1, 0, 0, -1$?

Answers: 3 Yes 1 No
Let $G$ be the vertex sum of $K_m$ and $K_n$. 
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1-connected Clique Sum

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$K_5 \oplus K_3$: 

mr($G$) = 2
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$$K_5 \oplus K_3:$$

$mr(G) = 2$

**Question**

Given two nonzero real numbers, $\lambda, \mu$ is there an $A \in S(G)$ such that the eigenvalues of $A$ are: $\lambda, \mu$, and 0 with multiplicity $n - 2$?
Theorem

Let $G$ be a connected graph and suppose there exist vertices $u, v$ of $G$ such that there is a unique path from $u$ to $v$ of length $d(u, v)$.

There is a unique path from $u$ to $v$ of length 2, so any $A \in S(G)$ must have 3 distinct eigenvalues.
Theorem

Let $G$ be a connected graph and suppose there exist vertices $u, v$ of $G$ such that there is a unique path from $u$ to $v$ of length $d(u, v)$. Then any $A \in S(G)$ has at least $d(u, v) + 1$ distinct eigenvalues.
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are $3, 2, 1, 0?$ $2, 1, 0, 0?$ $1, 1, 0, 0?$ $1, 0, 0, -1?$
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$\times$
Theorem

Let $G$ be a connected graph whose minimum rank is 2. Then we have the following restrictions in the Inverse Eigenvalue Problem:

- If $G$ is a vertex sum of two cliques, then a rank minimizing matrix for $G$ cannot have a nonzero eigenvalue of multiplicity two.
- If $G = K_k \lor K_\ell$, $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ must sum to 0.
- If $G = K_k \lor K_1$, $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ cannot sum to 0.

Any two nonzero eigenvalues not ruled out by the $mr^+ (G)$ or the restrictions above can be attained by a rank minimizing matrix.

$mr^+ (G)$ is the minimum rank over all positive semidefinite matrices in $S(G)$.
‘Solution’ of IEP for minimum rank 2 graphs

**Theorem**

Let $G$ be a connected graph whose minimum rank is 2. Then we have the following restrictions in the Inverse Eigenvalue Problem:

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- If $G = K_k \lor K_1$, where $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ cannot sum to 0.

Any two nonzero eigenvalues not ruled out by the $mr^+(G)$ or the restrictions above can be attained by a rank minimizing matrix.

$mr^+(G)$ is the minimum rank over all positive semidefinite matrices in $S(G)$. 
Theorem

Let $G$ be a connected graph whose minimum rank is 2. Then we have the following restrictions in the Inverse Eigenvalue Problem:

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- If $G = K_{k,\ell}$, $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ must sum to 0.

Any two nonzero eigenvalues not ruled out by the $mr^+$ of $G$ or the restrictions above can be attained by a rank minimizing matrix. $mr^+$ is the minimum rank over all positive semidefinite matrices in $S(G)$.
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- If $G = K_{k,\ell}$, $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ must sum to 0.
- If $G = K_{k,\ell} \lor K_1$, $k, \ell \geq 3$, then the two nonzero eigenvalues of a rank minimizing matrix for $G$ cannot sum to 0.

Any two nonzero eigenvalues not ruled out by the $\mr(G)$ or the restrictions above can be attained by a rank minimizing matrix. $\mr(G)$ is the minimum rank over all positive semidefinite matrices in $S(G)$. 

Wayne Barrett (BYU)
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Any two nonzero eigenvalues not ruled out by the $\text{mr}_+(G)$ or the restrictions above can be attained by a rank minimizing matrix.

$\text{mr}_+(G)$ is the minimum rank over all positive semidefinite matrices in $S(G)$. 
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $G$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more: If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$. 
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The theorem actually says more:
Inverse Eigenvalue Problem for Trees

Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $G$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
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The theorem actually says more:
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Let $G$ be a graph on $n$ vertices.
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1. Given $n$ distinct real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, is there a matrix $A \in S(G)$ with eigenvalues equal to $\lambda_1, \lambda_2, \ldots, \lambda_n$?
Let $G$ be a graph on $n$ vertices.

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2. Given a vertex $v$ of $G$ and $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, is there a matrix $A \in S(G)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$?
Open Questions

Let $G$ be a graph on $n$ vertices.

1. Given $n$ distinct real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, is there a matrix $A \in S(G)$ with eigenvalues equal to $\lambda_1, \lambda_2, \ldots, \lambda_n$?

2. Given a vertex $v$ of $G$ and $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, is there a matrix $A \in S(G)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$?

Intuition: Since both results are true for a tree, it seems they ought to be true for any connected graph.
Question 1 for $K_n$

Theorem

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then there exists an $A \in S(K_n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if $\lambda_1 > \lambda_n$.

$n = 2$: Let $A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix}$

$n > 2$: Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with $\lambda_1 > \lambda_n$

Case 1. $\lambda_2 = \lambda_n$:

Let $A = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_1 & \lambda_2 \end{bmatrix}$

$A$ has eigenvalues $\lambda_1$ and $\lambda_2$ with multiplicity $n-1$. 
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$n = 2$: Let $A = \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$
Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then there exists an $A \in S(K_n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if $\lambda_1 > \lambda_n$.

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Case 1. $\lambda_2 = \lambda_n$: 
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Case 1. $\lambda_2 = \lambda_n$: Let $A = \frac{\lambda_1 - \lambda_2}{n} J_n + \lambda_2 I_n$
Question 1 for $K_n$

**Theorem**

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then there exists an $A \in S(K_n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if $\lambda_1 > \lambda_n$.

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Case 1. $\lambda_2 = \lambda_n$: Let $A = \frac{\lambda_1 - \lambda_2}{n} J_n + \lambda_2 I_n$

$A$ has eigenvalues $\lambda_1$, and $\lambda_2$ with multiplicity $n - 1$. 
Case 2: $\lambda_2 > \lambda_n$
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By the induction hypothesis, there exists $B \in S(K_{n-1})$ with eigenvalues $\lambda_2, \ldots, \lambda_n$. 
Case 2: $\lambda_2 > \lambda_n$

By the induction hypothesis, there exists $B \in S(K_{n-1})$ with eigenvalues $\lambda_2, \ldots, \lambda_n$.

Since $K_{n-1}$ is connected, $\lambda_2 > b_{ii}$ for $i = 1, \ldots, n - 1$. 
Question 1 for $K_n$

Case 2: $\lambda_2 > \lambda_n$

By the induction hypothesis, there exists $B \in S(K_{n-1})$ with eigenvalues $\lambda_2, \ldots, \lambda_n$.

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Let $\begin{bmatrix} p & r \\ r & q \end{bmatrix} = Q^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & b_{11} \end{bmatrix} Q$
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Question 1 for $K_n$

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Let $G$ be a graph on $n$ vertices.

1. Given $n$ distinct real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, is there a matrix $A \in S(G)$ with eigenvalues equal to $\lambda_1, \lambda_2, \ldots, \lambda_n$?

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Let $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$. 
Question 2 for $K_n$

Let $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$.

By Duarte’s theorem we can choose $y \in \mathbb{R}^{n-1}$, $d \in \mathbb{R}$ so that the matrix

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Wayne Barrett (BYU)  
Inverse Eigenvalue Problem for Graphs  
October 13, 2011 16 / 22
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We used the fact that $G(A)$ is a star.
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We no longer need all the inequalities, just that $\lambda_1, \lambda_2, \ldots, \lambda_n$ is attainable for a star if $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues when the central vertex is deleted.
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More possibilities are attainable for a star than any other tree.
Let $Q$ be an orthogonal $n - 1 \times n - 1$ matrix.
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$$C = \begin{bmatrix} Q^T & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} M & y \\ y^T & d \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0^T & 1 \end{bmatrix}$$
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This matrix still has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. 
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This matrix still has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

and $Q^TMQ$ still has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. 

Now the question is how well can we control the zero/nonzero structure of $Q^TMQ$ and $Qy$?
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This matrix still has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

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Now the question is how well can we control the zero/nonzero structure of the matrices $Q^T M Q$ and $Qy$?
Question 2 for $K_n$

$$C = \begin{bmatrix} Q^T M Q & Q y \\ y^T & d \end{bmatrix}$$
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Suppose we let $Q = I - 2xx^T$ where $x$ is a unit vector in $R^{n-1}$
Question 2 for $K_n$

$$C = \begin{bmatrix} Q^T M Q & Q y \\ y^T & d \end{bmatrix}$$

Suppose we let $Q = I - 2x x^T$ where $x$ is a unit vector in $R^{n-1}$

A calculation gives:

$$(Q^T M Q)_{ij} = 2x_i x_j \left[ 2 \sum_{i=1}^{n-1} \mu_i x_i^2 - \mu_i - \mu_j \right]$$
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As long as $\mu_1 > \mu_{n-1}$, it is possible to make all of these nonzero.
Question 2 for $K_n$

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Then $C \in \mathcal{S}(K_n)$, has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$
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Question 2 for arbitrary graphs

Transfer theorems: A representative is
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Theorem

Let $T$ be a tree on $n > 2$ vertices, let $u,v$ be adjacent vertices of $T$, and let $w$ be any other vertex of $T$. Given any $2n-1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \ldots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there is a matrix $A \in S(G)$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(w)$. 
Transfer theorems: A representative is

**Theorem**

Let \( T \) be a tree on \( n > 2 \) vertices, let \( u, v \) be adjacent vertices of \( T \), and let \( w \) be any other vertex of \( T \). Let \( G \) be the graph obtained from \( T \) by inserting an edge between \( u \) and every vertex in \( N(v) \setminus \{u\} \) and between \( v \) and every vertex in \( N(u) \setminus \{v\} \).
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**Example:**

```
  u -- v
    ^   |
     ^  |
    1   2
```
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**Example:**

![Diagram](image-url)
Proof technique

Similarity by $Q \oplus I_{n-2}$ where $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$. 

So we can transfer Duarte's result to many other graphs.

A major limitation is that the vertices $u$ and $v$ become twins in the graph $G$. We do not have a general technique to transfer Duarte's theorem (or any of the other inverse eigenvalue theorems for trees) if the vertices are not twins.
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Problem Graphs

One encounters graphs without twins when the number of vertices is 5.

Don’t know the answers to questions 1 and 2 for these graphs.
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- C5
- House Graph
- Bull graph
- Gem Graph
- The Bull graph